University of Houston

~~String Theory~~

Quantum Theory of Flux Tubes

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1 A Brief History of Problems in Hadronic Physics

During the 1960s and early 70s (pre-QCD) one of the main challenges faced by the physics community was the huge proliferation of strongly interacting or confined states of particles known as hadrons. There existed hadronic resonances with exceptionally high spin. The spin J was related to the mass squared of the lightest particle as

$$J = \alpha' m^2 + \alpha(0) \tag{1}$$

where $\alpha' \sim 1(\text{GeV})^{-2}$ is called the universal Regge slope. Experimental results were tested up to $J = 11/2$ and was obvious that it would go on indefinitely see fig 1.

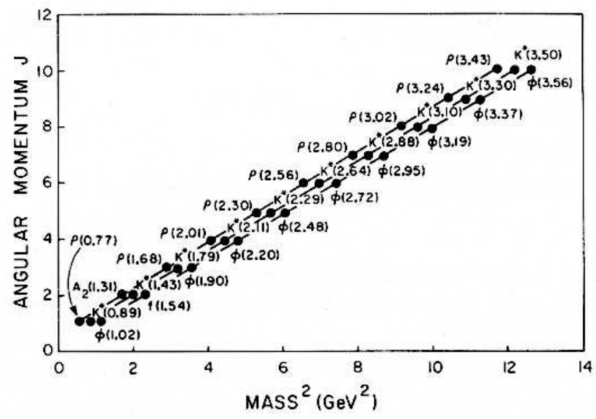


Figure 1: Regge Slope

Every point on the J vs M^2 plot is called a *resonance*. The huge number of resonances meant that they weren't fundamental at all. Renormalizable (consistent) QFTs were limited to spin-0, spin-1/2 and spin-1 with examples such as abelian gauge theories, scalar, Yang-Mills theory. This "restriction" to low-spin QFTs was successful because it described electromagnetic interactions with fundamental particles having spin-1/2 and spin-1. This was a robust framework for weak interactions but failed when strong interactions were naively treated like this.

Another puzzel was apparent when we consider the high-energy behavior of scattering amplitudes. Consider an elastic scattering process with incoming:(p_1, p_2) and outgoing:(p_3, p_4) momenta. Mass squared of the particle $m^2 = -p^2$ where the metric is mostly positive[1]. The Mandelstam variables are defined as

$$s = -(p_1 + p_2)^2, \; t = -(p_2 + p_3)^2, \; u = -(p_1 + p_3)^2 \tag{2}$$

and $s + t + u = \sum_i m_i^2$. The external legs (see fig 2) transform in the adjoint representation of the flavor group which for three flavors is $SU(3)$. Consider now a term in the scattering amplitude proportional to the group-theory factor $tr(\lambda_1\lambda_2\lambda_3\lambda_4)$. Where λ_i is the flavor matrix. This factor is invariant under the cyclic permutation $1234 \rightarrow 2341$, Bose statistics require that the corresponding amplitude should admit the symmetry $p_1p_2p_3p_4 \rightarrow p_2p_3p_4p_1$. We immediately see this in terms of the Mandelstam variables that $s \leftrightarrow t$ should be a symmetry of the amplitude $A(s, t)$. We know the leading contribution comes from tree-level terms in fig 2. *The challenge in building QFTs of particles of high spin is that tree level diagrams with exchange of high spin particles have a divergent behavior at high energies.* In fig 2 consider the interaction $\phi^*\phi\sigma$, where the exchange particle σ is spin-0 then

$$A(s, t) = -g^2/(t - M^2) \tag{3}$$

[1] diag$(- + +...+)$

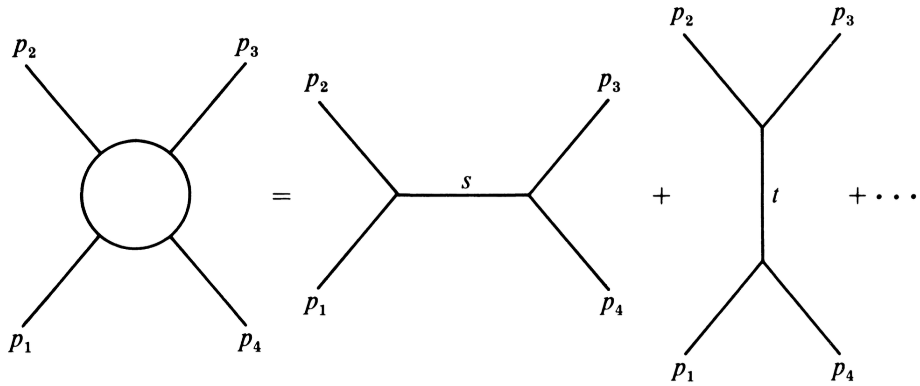


Figure 2: Tree Level scattering amplitude of Mesons

where g is the coupling constant and M is the mass of the *sigma* particle. In the limit $t \to \infty$ this amplitude vanishes. This is fantastic high energy behavior. Suppose the σ particle is now spin J field $\sigma_{\mu_1\mu_2\ldots\mu_J}$: The tree-level coupling will look like $\phi^* \partial_{\mu_1}\partial_{\mu_2}\ldots\partial_{\mu_J}\phi.\sigma^{\mu_1\mu_2\cdots\mu_J}$. This gives $2J$ factors of momenta. If the ϕ(s) are scalars then scattering amplitude of the exchange in the t channel will have the form

$$A_J(s,t) = -\frac{g^2(-s)^J}{t - M^2} \tag{4}$$

for high enough energies. For high Js, this amplitude is more and more divergent. What will happen if we try to sew the tree-level diagrams? See fig 3.

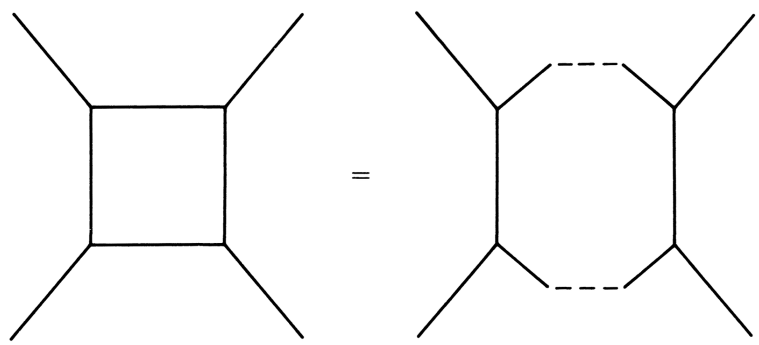


Figure 3: Sewed Diagrams

The loop integral in n dimensions will look like $\int d^n p A^2/(p^2)^2$. In 4 dimensions this integral is well behaved for $J < 1$, and unrenormalizable for $J > 1$. We know that there will be an exchange of strongly interacting particles of various masses and spins in the t-channel diagram. The general form of the amplitude is

$$A(s,t) = -\sum_J \frac{g_J^2(-s)^J}{t - M_J^2} \tag{5}$$

where if you notice we can now have the depndence on J of the couplings and the masses among other quantum numbers that we don't care about right now. If this is indeed a finite sum (in the high energy regime) then the most dominating contribution comes from the exchange particle with largest J. However this is not what is observed in nature. The high energy behavior is of hadron scattering amplitude is much smaller than any term in the series above. Also the assumption that this sum is finite is not possible because there is no such hadron which has the "largest" spin. As mentioned earlier we can go as high as we want as long we're on the Regge slope. When viewed as an infinite sum the high energy behavior is captured by the whole sum better than any individual term in the series. We also expect the amplitude to have both s and t-channel poles or neither. A finite sum on the other hand means, for fixed t, there are no s-channel poles. Which can be interpreted as an entire function of s, as long as there are only a finite number of terms in the sum. Due to this reason perturbative QFT satisfies crossing symmetry by including both s and t-channel diagrams. The point is we have to make peace with the amplitude being an infinite sum which is an entire function of s, with fixed t and it might diverge for finite values of s giving poles in the s-channel. This means that it is not important that s-channel diagrams should be included separately they're probably already in equation (5). A similar argument is valid if we start with s-channel poles with amplitude

$$A'(s,t) = -\sum_J \frac{g_J^2(-t)^J}{s - M_J^2}. \tag{6}$$

Imagine now choosing g_J and M_J such that $A(s,t) = A'(s,t)$. This means the entire amplitude can be expressed as a sum over either s-channel poles or t-channel poles. This is absurd since all our life we've learnt that we have to sum over both s and t-channel poles (diagrams). It was shown in 1968 by Dolen, Horn and Schmid that indeed the amplitudes are equal for small s and t. This was the first evidence of "Duality". Which meant that s- and t-channel diagrams give a 'dual' description of the same physics. The question that puzzled everyone was that is this "Duality" an approximation or a principle? A solution was found by Veneziano who postulated the following expression

$$A(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)) - \alpha(t))} \tag{7}$$

$$\Gamma(u) = \int_0^\infty t^{u-1}e^{-t}dt \text{ (Euler Gamma Function)}^2 \tag{8}$$

and $\alpha(s)$ is the 'Regge trajectory'. It was postulated that $\alpha(s) = \alpha(0) + \alpha'(s)s$; which are the Regge intercept and the Regge slope respectively. Using one of the properties we can write it in the following form

$$\Gamma(u) = \frac{\Gamma(u+1)}{u} \tag{9}$$

$$\Gamma(u) = \frac{\Gamma(u+n)}{u(u+1)...(u+n-1)}$$

The above equation tells us that Γ is singular only at simple poles $u = 0. -1, -2...$ The behavior near $-n$ is therefore

$$\Gamma(u) \sim \frac{1}{u+n}\frac{(-1)^n}{n!} \tag{10}$$

Consider now a more closely related Euler Beta function

$$B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)} \tag{11}$$

This is related to the Veneziano amplitude if $u = -\alpha(s)$ and $v = -\alpha(t)$. *Recall the fact that the only poles allowed in tree level diagrams in QFTs are simple poles and the residue must be a polynomial.* Using the above fact and the properties of these functions we express the Veneziano amplitude as

$$A(s,t) = -\sum_{n=0}^\infty \frac{(\alpha(s)+1)(\alpha(s)+2)...(\alpha(s)+n)}{n!}\frac{1}{\alpha(t)-n} \tag{12}$$

[2]Important Properties of the Gamma function $\Gamma(u+1) = u\Gamma(u),\ \Gamma(1) = 1, \Gamma(u) = (u-1)!$

Invoking the following symmetry $A(s,t) = A(t,s)$ we have

$$A(s,t) = -\sum_{n=0}^{\infty} \frac{(\alpha(t)+1)(\alpha(t)+2)...(\alpha(t)+n)}{n!} \frac{1}{\alpha(s)-n} \tag{13}$$

Now for a simple 'Regge trajectory' given by $\alpha(t) = \alpha' t + \alpha(0)$ the singularities of (12) are simple poles corresponding to t-channel exchange of particles of mass $M^2 = (n - \alpha(0)/\alpha')$ where $n = 0, 1, 2, ...$ This means that particles of mass $n - \alpha(0)/\alpha'$ have spin at most n. In other words the smallest possible mass of particle of spin J is thus $(J - \alpha(0))/\alpha'$. Which makes it clear that particles of mass M^2 lie on the 'Regge trajectory'. We will be interested in a positive slope ($\alpha' > 0$) since otherwise the particles would have negative mass (tachyons). The equality of (12) and (13) is the major result of a 'duality'. It is important to remember that in order to preserve unitarity and the absence of ghosts in a theory, the residues of poles must be positive in a QFT which is not obvious in the above equations for amplitudes. This led to the 'no-ghost theorem' which basically means that ghosts are absent if there are some restrictions on the 'Regge intercept' and the spacetime dimension[3]. Along with the above requirement we also have to understand the high energy behavior of the Veneziano Amplitude. The physical region for elastic scattering is positive s and negative t or vice-versa. I would suggest the reader to see [1] to understand how we check this. The fact of the matter is that the Veneziano Amplitude was considered an ad hoc way to deal with this 'crossing-symmetry'. This however intrigued many physicists at the time and eventually led to the birth of many mathematical advancements and of the Bosonic String Theory (in 26 dimensions) and the Superstring Theory (in 10 dimensions).

1.1 A Strongly Interacting Story

Most progress is physics has largely come from trying to understand a particular puzzle only to find out that uncovering those mysteries more often than not leads to many unrelated and sometimes more profound results. Understanding the theory of strong interactions has a similar story for what I'll be discussing in the rest of this report.
The physics of strong interactions has the following features:

- The fundamental quarks are confined into hadrons.
- The strength of the interaction increases as the separation between the quarks increases.
- This theory of Strong Interactions is called it QCD. It is a gauge theory (gluons) coupled with matter (quarks) with local symmetry. They transform as

$$A^{ab}_{\mu}(x) \to A^{ab}_{\mu}(x) + \partial\mu\epsilon^{ab}_{\mu}(x) + ..., \quad \psi^a(x) \to \psi^a(x) + i\epsilon^{ab}(x)\psi^b \tag{14}$$

- Symmetry Group $SU(3) \times SU(2) \times U(1)$
- Lagrangian

$$\mathcal{L} = \frac{1}{2g^2_{YM}}(\partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]^2) \tag{15}$$

  has all the gauge interaction information. The "constant" in front g_{YM} is the strength of the interaction and runs with the energy scale of the theory. We know that g_{YM} is small at very high energies (asymptotic freedom) and is very strong at low everyday energies. This means that at low energies we can't use standard perturbative diagrammatic expansion techniques. So how do we understand confinement and the spectrum of Hadrons?

[3] As we will see later that Spacetime dimension = 26 and $\alpha(0) = 1$.

1.2 A Quick Comment on Quantum Gravity(QG):

Such a 'bad' UV behavior was also observed when physicists attempted to quantize gravity. In general relativity the gravitational field is a massless spin−2 field called the graviton field. Its interactions are governed by the nonabelian local symmetry group called diffeomorphisms of spacetime. Despite the similaties in the two theories, Yang-Mills and QG are quite significantly different due to the different spins of fundamental objects. For a t-channel exchange of a particle

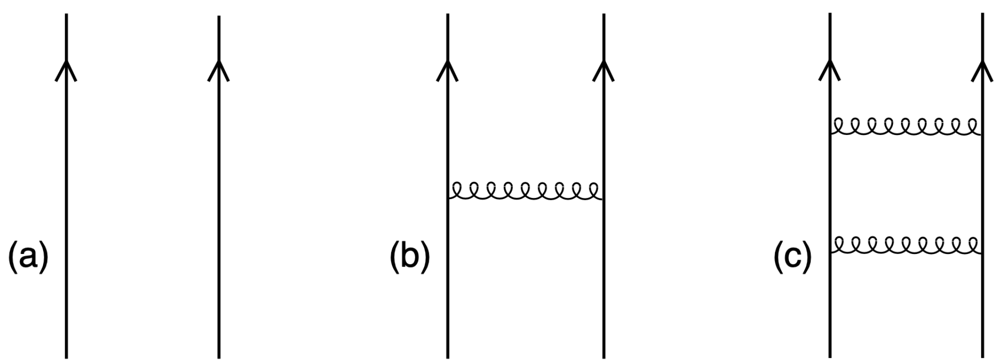
$$A(s,t)_{YM} \sim s/t \tag{16}$$

In four dimension this behavior is barely renormalizable. However for QG

$$A(s,t)_{QG} \sim s^2/t \tag{17}$$

this is crazy and hopelessly unrenormalizable. The "Veneziano" type also called dual models always predicted massless particles. For example the $s = 0,\ t = 0$ poles and $\alpha(0) = 1$. These can also be of various spins. In particular a massless spin−2 particle. Its interaction behavior is similar to those in general relativity. These dual models, which were on the verge of extinction gained a lot of virtue due to this fact[4]. Therefore Dual models in higher dimensions are not just consistent theories describing but theories of all fundamental interactions.

1.2.1 A Dimensional Analysis Explanation



If you look at fig above it shows two propagating particles with one and two graviton exchanges. One graviton exchange is proportional to G_N (Newton's const.). If we calculate the ratio of this one-graviton-exchange amplitude to the original one we see that it must proportional to the dimensionless combination given by $G_N E^2 \hbar^{-1} c^{-5}$ where E of course is the energy scale that characterizes the process. This is literally "the only" dimensionaless combination that is possible. For $\hbar = c = 1$ we can then define Planck Mass $M_p = G_N^{-1/2} = 1.22 \times 10^{19}$ GeV and Planck Length $M_p^{-1} = 1.6 \times 10^{-33} cm$. The ratio mentioned above in terms of Planck mass and E is $(E/M_p)^2$. If you notice carefully, what this means is that the interaction coupling for such a process is *irrelevant* i.e., becomes weaker a low energy scales (even at particle physics energy scales). Whereas when $E > M_p$ it becomes relevant and perturbation theory breaks down. Now if we look at two-graviton-exchange we see that there is a sum over intermediate states. Let the energy of those be E' then the ratio with the zero-graviton exchange is

$$G_N^2 E^2 \int dE' E'$$

[4]The Reader can also look at Kaluza-Klein Theory and Supersymmetry for some further developments of this argument. I will not be explicitly talking about these here, unless specified otherwise to give some context.

which obviously diverges at arbitrary high energies. In position space we that this problem comes when all graviton vertices become coincident. Divergence grows with each graviton. In other words, at shorter distances there exist more gravitons which also exchange gravitons with one another and this diverges at shorter and shorter scales. This is the non-renomalizability problem.

2 A Heuristic Story

2.1 A Hint in QCD

QCD Lagrangian

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\sum_{q=1}^{N_f}\bar{\psi}_q^i(\not{D}_\mu)_{ij}\psi_q^j - \sum_{q=1}^{N_f} m_q\bar{\psi}_q^i\psi_q^i \tag{18}$$

where

$$\begin{aligned} F_{\mu\nu} &\equiv \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu] \\ (D_\mu)_{ij} &\equiv \delta_{ij}\partial_\mu + ig\frac{\lambda_{ij}^a}{2}A_\mu^a \\ A_\mu &\equiv A_\mu^a\frac{\lambda_a}{2}, \;\; a = 1, ..., 8 \end{aligned} \tag{19}$$

The quark fields ψ_q^j are of flavour $q = 1, ..., 6$, color $j = 1, ..., 3$, m_q the current quark masses and λ^a are the Gell-Mann matrices of $SU(3)$. We have Beta function equation for the running coupling

$$\frac{d\alpha_s(\mu)}{d\log\mu^2} \equiv \beta(\alpha_s) = -\alpha_s\left(\beta_0(\frac{\alpha_s}{4\pi}) + \beta_1(\frac{\alpha_s}{4\pi})^2 + ...\right) \tag{20}$$

where $\alpha_s \equiv g^2/4\pi$ and $\beta_0 = 11 - \frac{2}{3}N_f$. This tells us that we can only do perturbation theory when $\mu >> \Lambda_{QCD}$ (The confinement scale). It also tells us that bound-quark-states occur at energy scales 1 fm or Λ_{QCD}^{-1} which is also the average Hadronic size. Due to this confinement feature quark masses aren't physical which means they can't be directly measured. Therefore we introduce constituent quark masses which contain the current mass and corrections due to confinement. In such a scenario light mesons and baryons obey Regge trajectories that we talked about earlier. Since no free quarks have been detected it is inferred that the interaction among them has to be strong a long length scales and that $q\bar{q}$ pair is created when the quarks are considerably separated. They always appear in this hadronic state of mass $1GeV$ at the separation of 1 fm. What this suggests is that there exists a linear density between a quark and an anti-quark of order

$$T = \frac{\Delta E}{\Delta r} \simeq 1\frac{GeV}{fm} \simeq 0.2GeV^2 \tag{21}$$

See fig This resembles a hadron with two quarks at the ends of a string. The fields lines, as opposed to spreading out like in Electrodynamics, confine themselves into a single tube called the chromoelectric flux tube. Understanding then the light meson spectrum implies that we solve a relativistic many-body bound state problem where confinement is related to spontaneous chiral symmetry breaking.

As a result at short distances ($<$ 1 fermi), the quark-anti-quark potential is Coulombic, due to asymptotic freedom. At large distances we expect the potential to be linear due to the formation of confining flux tubes. If the tubes are longer than they are thick we can describe them using strings. We can invoke a semi-classical treatment of these strings [5] which gives us the following potential

$$V(r) = Tr + \mu + \frac{\gamma}{r} + O(1/r^2) \tag{22}$$

This tells us that objects like strings naturally exist in QCD however the quantization of the highly quantum QCD string which could reproduce the light meson and the glueball spectrum is a much harder problem.

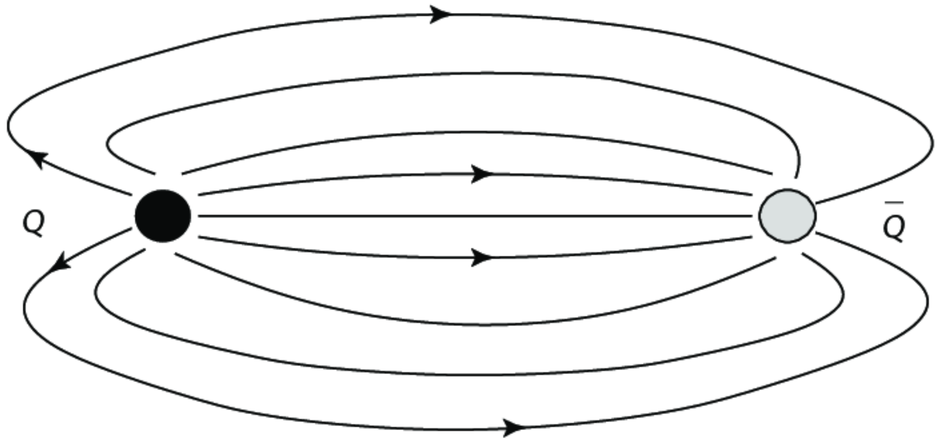


Figure 4: Flux Tubes confining themselves into a thin tube as the quark-antiquark distance increases.

3 A Short Course on String Theory

One possible[5] way to remove the divergences and ensure that the story remains consistent is *String Theory*. What can one expect to find when they try to construct a Quantum Field Theory of one dimensional objects?

- Every such theory has a massless spin-2 state whose interactions at low energy reduces to general relativity.
- A perturbative quantum gravity theory.
- "Grand Unification", we find gauge groups large enough to contain the one describing the standard model.
- Extra Dimensions
- Supersymmetry
- Chiral Gauge Couplings
- No free parameters
- Uniqueness

Let our fundamental object be a 1-dimensional mathematical curve[6]. It is possible that the string can be open (with endpoints) and closed (topologically circles). We choose the coordinate $\sigma \in [0, \pi]$ and to describe evolutionary dynamics add a timelike parameter τ, see fig 5. When the string propagates in time it "forms" a world-sheet which is obviously the generalization of the world line for a point particle. The string moves in D flat spacetime dimesnions with metric $\eta_{\mu\nu} = diag(-, +, +, ..., +)$. We first consider a point particle see appendix A. The string will look like $X^\mu(\tau, \sigma)$. We naturally demand that the action describing this is free of the parameters but depends on the embedding in spacetime. This is necessary for a

[5] Perhaps the best possible way.

[6] A natural question that should come to mind is why stop there? Why not higher dimensional objects. The answer is that if the fundamental objects are considered to be of any dimension greater than one, even though we remove the divergences discussed above, we encounter other divergences coming from internal degrees of freedom. One dimensional strings seem to be a unique way to control most divergences. Also this section is to mostly appreciate the formulation of String Theory but is not very important for the bigger picture of the review

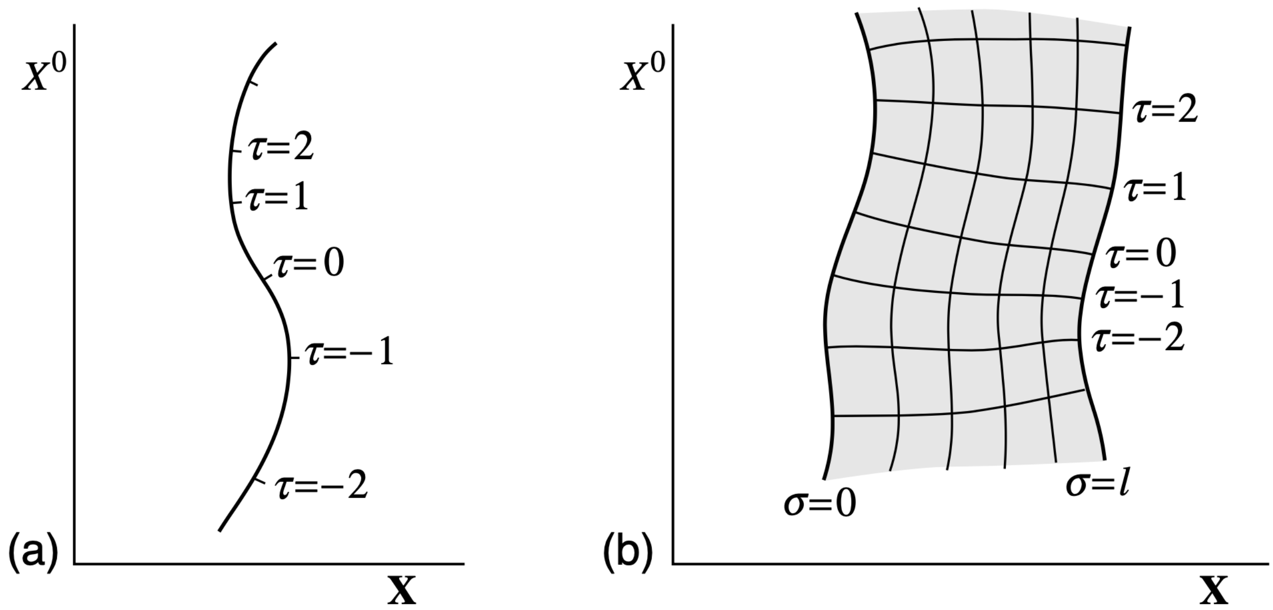


Figure 5: World line and world sheet parametrization.

relativistic quantum theory. For point particles we extremize the world-line so for a one-dimensional string we extremize the world-sheet see appendix B. Notice the actions were constructed with the motivation that we need simple actions to accommodate a certain set of symmetries. But simplicity is not the best way to go about this. Any consistent quantum theory should be formulated by the underlying symmetries. So after some "magic"[7] the most general (diff×Weyl)−invariant action

$$S'_P = -\int_M d\tau d\sigma(-\gamma)^{1/2}\left(\frac{1}{4\pi\alpha'}\gamma^{ab}\partial_a X^\mu \partial_b X_\mu + \frac{\lambda}{4\pi}R\right) \tag{23}$$

where R is the Ricci scalar constructed from the world-sheet metric. For the purposes of our discussion and the point that I want to make lets shift gears and discuss the scenario when the string has endpoints.

3.1 Open String Spectrum

Consider Light-cone coordinates (boosting in a direction)

$$x^{\pm} = 2^{-1/2}(x^0 \pm x^1), \quad x^i, \; i = 2, ..., D-1 \tag{24}$$

Set $\tau = x^+$, p^- is Energy and the rest (x^-, p^+) − logitudnal, (x^i, p^i) − transverse are spatial coordinates. For open strings $-\infty < \tau < +\infty$ and $0 \le \sigma \le l$. See fig 5. In the light-cone gauge

$$\begin{aligned} X^+ &= \tau \\ \partial_\sigma \gamma_{\sigma\sigma} &= 0 \\ det\gamma_{ab} &= -1 \end{aligned} \tag{25}$$

The Lagrangian becomes

$$L = -\frac{l}{2\pi\alpha'}\gamma_{\sigma\sigma}\partial_\tau x^- + \frac{1}{4\pi\alpha'}\int_0^l d\sigma\left(\gamma_{\sigma\sigma}\partial_\tau X^i \partial_\tau X^i - \gamma_{\sigma\sigma}^{-1}\partial_\sigma X^i \partial_\sigma X^i\right) \tag{26}$$

[7]Some conditions that I call magic here but the reader can see [3]

Momentum density conjugate to $X^i(\tau,\sigma)$ is

$$\Pi^i = \frac{\delta L}{\delta(\partial_\tau X^i)} = \frac{1}{2\pi\alpha'}\gamma_{\sigma\sigma}\partial_\tau X^i = \frac{p^+}{l}\partial_\tau X^i \tag{27}$$

The Hamiltonian is

$$H = \frac{l}{4\pi\alpha' p^+}\int_0^l d\sigma \left(2\pi\alpha'\Pi^i\Pi^i + \frac{1}{2\pi\alpha'}\partial_\sigma X^i\partial_\sigma X^i\right) \tag{28}$$

which is the Hamiltonian for $D-2$ free fields X^i, with p^+ or the Energy being conserved. For the boundary condition $\partial_\sigma X^i = 0$ at $\sigma = 0.l$, we can write the solution to the wave equation given by the Hamilton's equatins of motion [3] implying the wave equation $\partial_\tau^2 X^i = c^2\partial_\sigma^2 X^i$, whose solution is

$$X^i(\tau,\sigma) = x^i + \frac{p^i}{p^+}\tau + i(2\alpha')^{1/2}\sum_{n=-\infty,n\neq 0}^{\infty}\frac{1}{n}\alpha_n^i \exp\left(-\frac{\pi inc\tau}{l}\right)\cos\left(\frac{\pi n\sigma}{l}\right) \tag{29}$$

Quantization requires us to impose equal time commutation relations

$$\begin{aligned}[][x^-, p^+] &= -i \\ [X^i(\sigma), \Pi^j(\sigma')] &= i\delta^{ij}\delta(\sigma-\sigma')\end{aligned} \tag{30}$$

while any other combination vanishes. Similarly in terms of Fourier components

$$\begin{aligned}[][x^i, p^i] &= i\delta^{ij} \\ [\alpha_m^i, \alpha_n^j] &= m\delta^{ij}\delta_{m,-n}\end{aligned} \tag{31}$$

The modes, for each m and i satisfy a harmonic oscillator algebra

$$\alpha_m^i \sim m^{1/2}a, \;\; \alpha_{-m}^i \sim m^{1/2}a^\dagger, \;\; m>0 \tag{32}$$

where $[a, a^\dagger] = 1$. Direction of oscillation i and the harmonic m, label the oscillator. A state $|0;k\rangle$ where $k = (k^+, k^i)$ will be annihilated by the lowering operator and will be an eigenstate of the canter-of-mass momenta

$$\begin{aligned} p^+|0;k\rangle &= k^+|0;k\rangle, \;\; p^i|0;k\rangle = k^i|0;k\rangle, \\ \alpha_m^i|0;k\rangle &= 0, \;\; m>0\end{aligned} \tag{33}$$

Any general state can be built with raising operators

$$|N;k\rangle = \left[\prod_{i=2}^{D-1}\prod_{n=1}^{\infty}\frac{(\alpha_{-n}^i)^{N_{in}}}{(n^{N_{in}}N_{in}!)^{1/2}}\right]|0;k\rangle \tag{34}$$

States are labeled by center-of-mass momenta (k^+, k^i) and the occupation number N_{in} for each mode (i,n). One can think of the center-of-mass momenta as the degree of freedom for the point particle and the oscillator part represents the internal degrees of freedom. So every choice of the occupation number corresponds to a different particle (or spin state). These states form a Hilbert space $\mathcal{H}_1$, of one open string. Therefore the state $|0;0\rangle$ is the ground state of a single string with zero momentum and not to be confused with with the zero-string vacuum state. So remember that the operators above don't create or destroy strings, they act within the the space of states of a string. For n strings, the Hilbert space will be n copies of the above space. Hence, the full Hilbert space "in the free limit" is

$$\mathcal{H} = |vacuum\rangle \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus ... \tag{35}$$

Finally the Hamiltonian with the mode expansion will be

$$H = \frac{p^i p^i}{2p^+} + \frac{1}{2p^+\alpha'}\left(\sum_{n=1}^{\infty} \alpha^i_{-n}\alpha^i_n + A\right) \tag{36}$$

where A is an unknown constant from the commutators. I will not be discussing the way to determine this constant (the reader can see [3]) but what one finds is that the theory admits all physical symmetries in the special case when $A = -1$ and the spacetime dimension is precisely $D = 26$. In order to preserve Lorentz and Weyl invariance we get

$$A = \frac{2-D}{24} \tag{37}$$

this is an example of Casimir energy, due to the fact that the string has a finite length. Let us now look at the lightest string state is

$$|0;k\rangle\,, \quad m^2 = \frac{2-D}{24\alpha'} \tag{38}$$

notice for $D > 2$, the mass squared is negative. this state is a tachyon. Recall the potential energy for a scalar field is $\frac{1}{2}m^2\phi^2$. The negative mass-squared implies an unstable vacuum[8]. By exciting $n = 1$ we get the lowest excited states

$$\alpha^i_{-1}|0;k\rangle\,, \quad m^2 = \frac{26-D}{24\alpha'} \tag{39}$$

Now lets break this down. Lorentz invariance requires a specific value of D. For massless particles there is no rest frame. Lets choose frame $p^\mu = (E, E, 0, ..., 0)$. The $SO(D-2)$ acting on transverse directions leaves p^μ invariant, the internal states form a representation of this smaller group. Massless particles are labelled by helicit λ, which is the eigenvalue under the single $SO(2)$ generator. Therefore from Lorentz invariance we only need one state. However CPT symmetry takes $\lambda \to -\lambda$ so that means two states for $\lambda \neq 0$. In D dimensions a massless vector has $D-2$ spin states. For $n = 1$ we got $D-2$ states $\alpha^i_{-1}|0;k\rangle$ implies that they must be mass less which means $D = 26$ and $A = -1$. The quantum mechanical spectrum is Lorentz invariant for this specific number of spacetime dimensions. The theory, classically, is invariant under any D, but when quantized there is an anomaly except when $D = 26$. AT level N, the maximum eigenvalue of a given spin component, is N. For some fixed spin l,

$$l \leq \frac{D-2}{24} + \alpha' m^2 \tag{40}$$

where α' is our friend "The Regge Slope".

4 Coming Full Circle: AdS/CFT

The idea "in a way" came from the never-ending love for looking for a small parameter around which an expansion can be performed aka perturbation theory. We have already explained above that for QCD perturbation theory is not possible in the strongly coupled regime. t' Hooft suggested that gauge theories with the gauge group $SU(N)$ simplify when $N \to \infty$. The string coupling constant in this case is $1/N$. Then it is obvious that if $N = 3$ is similar to $N \to \infty$, then the Regge trajectory similarity can be explained. Maldacena in his 1997 paper [6] pointed out a correspondence between string theories and the large N limit of field theories. He observed that string theory in some backgrounds, including gravity is dual to a field theory. He looked at supersymmetric conformally invariant field theories in four dimensions. The conformal group in four dimensions is $S0(4,2)$ this includes Poincaré, scale and special conformal transformations. For an equivalence to exist these symmetries should also be realized in the dual theory. Locally, this symmetry group can only be realized in the five dimensional Anti-de-Sitter space (AdS_5). Supersymmetric strings move in ten dimensions, and the gauge theory has $SU(4) \simeq SO(6)$ global

[8] Let's ignore this instability for now.

symmetry. So we can say that $\mathcal{N} = 4$ $U(N)$ Yang-Mills is dual to ten dimensional superstring theory on $AdS_5 \times S^5$. The striking difference between this approach and the previous string revolutions is that strings are moving on a curved spacetime (AdS) which has a boundary at spatial infinity. A light-ray can go to the boundary and return in finite time even though the boundary is at spatial infinity. However, massive particles can never go to the boundary. The radius of curvature depends on N, this implies that in the large N limit, we can make the curvature as small as we want. The theory in AdS includes gravity because any string theory has gravity automatically built into it. This mapping between a gravitational theory and a field theory is highly non-trivial since the field theory lives in a lower dimension. In a way the field theory lives on the boundary of AdS spacetime. Notice that theories in which gravity is being treated quantum mechanically are topological since the integral is performed over all metrics. Hence, the metric dependence goes away. Also remember that when we say a theory includes gravity what we mean by that is a finite energy excitation in AdS space. Therefore we really sum over all spacetimes. Generally when discussing theories of quantum gravity we want asymptotic flatness but we don't rule out possibilities of spacetime having any topology as long it is asymptotically flat. So asymptotically AdS and asymptotically flat are special cases where one can choose a natural time and associated Hamiltonian to define the quantum theory. The presence of Black Holes means that this time coordinate is not globally well defined but certainly well-defined at infinity. Therefore keeping all these constraints in mind we can say that YM theory gives a non-perturbative definition of String theory on AdS in the large N limit. The field theory living in a lower spatial dimension also ensures that the theory of quantum gravity is Holographic.

4.1 Large-N

At first glance it may seen a little counter intuitive that by increasing the number of fields the theory would simplify instead of getting more complicated but when you realise that these fields are related by a symmetry then the collective behaviour gets constrained. Consider the $SU(N)$ Yang-Mills action

$$S_{YM} = -\frac{1}{2g^2} \int d^4x \, \mathrm{tr} \, F^{\mu\nu} F_{\mu\nu} \tag{41}$$

We know that

$$\Lambda_{QCD} = \Lambda_{UV} \exp\left\{ -\frac{3}{22} \frac{(4\pi)^2}{g^2 N} \right\} \tag{42}$$

where Λ_{QCD} is physical and Λ_{UV} is the introduced cut-off. If we naively take $N \to \infty$ limit for fixed g^2 and cut-off, then we don't get any parametric separation between the two-scales. However if we define the t'Hooft coupling first

$$\lambda = g^2 N \tag{43}$$

and now for fixed cut-off and λ we take the large N limit this ensures the fixing of Λ_{QCD}. So the new action is

$$S_{YM} = -\frac{N}{2\lambda} \int d^x \, \mathrm{tr} \, F^{\mu\nu} F_{\mu\nu}. \tag{44}$$

4.1.1 Large N Feynman Diagrams

Each gluon field is now an $N \times N$ matrix,

$$(A_\mu)^i_j, \ \ i, j = 1, ..., N, \tag{45}$$

the propagator index structure

$$\langle A^i_{\mu j}(x) A^k_{\nu l}(y) \rangle = \Delta_{\mu\nu}(x - y) \left(\delta^i_l \delta^k_j - \frac{1}{N} \delta^i_j \delta^k_l \right) \tag{46}$$

where $\Delta_{\mu\nu}$ is the usual photon propagator for a single gauge field. The $1/N$ factor comes from working with traceless $SU(N)$ gauge fields, rather than $U(N)$ gauge fields. We see a clear $1/N$ suppression so at leading order this term can be dropped.

$$\langle A^i_{\mu j}(x) A^k_{\nu l}(y)\rangle = \Delta_{\mu\nu}(x-y)\delta^i_l \delta^k_j \tag{47}$$

this ensures that we're really working with a $U(N)$ theory and not $SU(N)$. The two indices ($\bar{N} and N$) representation can be diagrammatically shown in fig 6. Arrows represent complex conjugate representations.

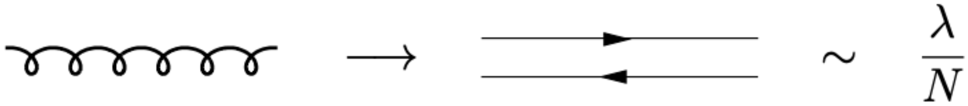


Figure 6: Double line propagator.

Cubic and Quartic vertex fig 7,8. Notice when we evaluate Feynman diagrams we can have a double ex-

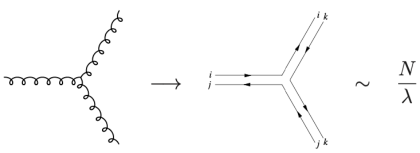


Figure 7: Cubic Vertex

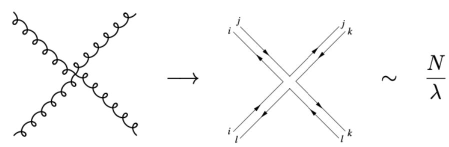
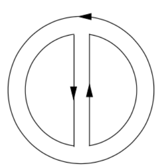


Figure 8: Quartic Vertex

pansion both in λ and in N. Generally any

$$\text{diag} \sim \left(\frac{\lambda}{N}\right)^{\#\text{propagators}} \left(\frac{N}{\lambda}\right)^{\#\text{vertices}} N^{\#\text{index contractions}} \tag{48}$$

Let's look at some examples. Consider a vacuum bubble 9 3 propagators, 2 vertices and 3 contractions



$$\sim \left(\frac{\lambda}{N}\right)^3 \left(\frac{N}{\lambda}\right)^2 N^3 \sim \lambda N^2$$

Figure 9: Double line vacuum bubble

give the final contribution of the diagram. When trying different contraction one easily comes to the conclusion that the dominant diagrams are the ones that can be drawn on a plane. Therefore these are called *Planar Diagrams.* Now this result is amazing because we don't have to care about all Feynman diagrams but just the ones that can be drawn on a plane.

4.1.2 Topology of Diagrams

Another insight is that these planar diagrams can be drawn in the surface of a sphere. A feynman diagram can tile a 2-dim surface Σ using the following map

$$\begin{aligned} E &= \#\text{of edges} = \#\text{of propagators} \\ F &= \#\text{of faces} = \#\text{of index loops} \\ V &= \#of vertices \end{aligned} \tag{49}$$

Therefore from before we get that the

$$\text{diag} \sim N^{F+V-E}\lambda^{E-V}. \tag{50}$$

We also know that we can characterize the Riemann surface using the Euler Character

$$\chi(\Sigma) = F + V - E = 2 - 2H \tag{51}$$

where H is the *genus*. The purpose of this small exercise was that the sum of Feynman diagrams are weighted by their topology

$$\text{diag} \sim N^{\chi}\lambda^{E-V}. \tag{52}$$

For each *genus*, we can have a different tiling of the Riemann surface in the t' Hooft expansion.

4.2 Engineering our QCD

I would like to conclude this review by outlining a qualitative step-wise approach to go about constructing a pure QCD[9]. The reader can see [7] to appreacite the full glory of this complicated procedure. Now we want to stick to non-supersymmetric gauge theories that exhibit confinement and asymptotic freedom. Consider QCD without matter fields. in p dimensions using dual superstring descriptions.

1. Start with $\mathcal{N} = 4$ Super Yang Mills in four dimensions.

2. Compactify this theory on $\mathbb{R}^3 \times \mathbb{S}^1$ to get a non-supersymmetric three-dimensional theory with anti-periodic boundary conditions for fermions around the circle. Boundary conditions break SUSY and at small radius of the circlue, fermions decouple from the theory since there are no zero frequency modes.

3. In the IR we are only left with gauge field degrees of freedom (pure QCD).

4. Do the same procedure in dual superstring theory. Take type IIB superstring theory on $AdS_5 \times S^5$.

5. Gauge coupling g_4 is related to the string coupling constant $g_4^2 \simeq g_s$. In the t' Hooft limit when $g_4^2 N \simeq g_s N$ is fixed $g_s \to 0$. Also $g_s N \gg 1$ implies that AdS curvature is small and string theory is approximated by classical supergravity. The metric becomes of a Euclidean black hole

$$ds^2 = \alpha' \sqrt{4\pi g_s N} \left(u^2(h(u)d\tau^2 + \sum_{i=1}^{3} dx_i^2) + h(u)^{-1}\frac{du^2}{u^2} + d\Omega_5^2 \right) \tag{53}$$

   where τ parametrizes the compactifying circle and

$$h(u) = 1 - \frac{u_0^4}{u^4}. \tag{54}$$

   The $i = 1, 2, 3$ corresponds to the three directions. The horizon of the geometry is at $u = u_0$ where $u_0 = 1/2R_0$.

[9]with minimal explanation.

6. This R_0, from the QCD point of view provides the UV cut-off scale.

7. The limit in which classical gravity is valid $g_s N \gg 1$, is the limit in which mass scale of QCD is much larger than cut-off scale $1/R_0$. When comparing this technique to Lattice QCD, we see that we have the full Lorentz invariance in three coordinates.

8. We see that the gauge coupling has dimensions of mass and it provides the a scale for the classical theory. Therefore the effective dimensionless parameter at a length scale l is lg^2 which goes to zero as $l \to 0$. This resembles asymptotic freedom. At large l coupling becomes large and we get a potential that looks like $V(r) \sim r$.

5 References

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Appendix

A What can we learn from point particles?

Consider first a point particle in D dimensions and its motion in spacetime is described by D function $X^{\mu}(\tau)$, where τ is a parameter along particle's worldline. Simplest Poincaré invariant action independent of the parametrization is

$$S = -m \int d\tau (-\dot{X}^{\mu} \dot{X}_{\mu})^{1/2} \tag{55}$$

where the derivative is with respect to τ. Varying this action δS gives us the expression for the D-velocity

$$u^{\mu} = \dot{X}^{\mu}(-\dot{X}^{\nu} \dot{X}_{\nu}) \tag{56}$$

$\dot{u}^{\mu} = 0$ represents free motion. Another useful form can be obtained by introducing an independent world-line metric $\gamma_{\tau\tau}(\tau)$ or the tetrad $\eta(\tau) = (-\gamma_{\tau\tau}(\tau))^{1/2}$. We get the action

$$S' = \frac{1}{2} \int d\tau \left(-\eta^{-1} \dot{X}^{\mu} \dot{X}_{\mu} - \eta m^2 \right) \tag{57}$$

and both the actions are Poincaré invariant and world-line reparametrization invariance[10]. The reason behind formulating S' is that it is path-integral friendly since we see the action is quadratic in derivatives.

B Constructing the Action

The Nambu-Goto action

$$S_{NG} = \int_M d\tau d\sigma - \frac{1}{2\pi\alpha'} (-det\ h_{ab})^{1/2} \tag{58}$$

where $h_{ab} = \partial_a X^{\mu} \partial_b X_{\mu}$ and $a, b \in \{\tau, \sigma\}$ is defined as the induced metric. The integral is taken over the world-sheet M. α' which has units $[L]^2$, is the Regge Slope and string-tension $T = 1/2\pi\alpha'$. To again avoid the derivates in the square-root we do the same trick of introducing a metric $\gamma_{ab}(\tau, \sigma)$ on the world-sheet this time. Taking the Lorentzian signature for this we have the Polyakov action

$$S_P[X, \gamma] = -\frac{1}{4\pi\alpha'} \int_M d\tau d\sigma (-\gamma)^{1/2} \gamma^{ab} \partial_a X^{\mu} \partial_b X_{\mu} \tag{59}$$

where $\gamma \equiv det\ \gamma_{ab}$. This action admits Diffeomorphism invariance, D-dimensional Poincaré invariance, Two-dimensional Weyl invariance[11]. The variation with respect to the metric gives the energy-momentum tensor

$$\begin{aligned} T^{ab}(\tau, \sigma) &= -4\pi(-\gamma)^{-1/2} \frac{\delta}{\delta \gamma_{ab}} S_P \\ &= -\frac{1}{\alpha'} \left(\partial^a X^{\mu} \partial_b X_{\mu} - \frac{1}{2} \gamma^{ab} \partial_c X^{\mu} \partial_c X_{\mu} \right) \end{aligned} \tag{60}$$

Diff invariance and Weyl Invariance implies $\boldsymbol{\nabla}_a T^{ab} = 0$ and $T^a_a = 0$ repectively. If you notice carefully the above actions S_P and S_{NG} basically describe two-dimensioanl field theories on the world-sheet. When calculating scattering amplitudes we can also see hat the matrix elements are given by two-dimensional QFT on the world-sheet. Therefore on the (if we ignore the index μ for now) world-sheet diffeomorphism invariance $X'^{\mu}(\tau', \sigma') = X^{\mu}(\tau, \sigma)$ tells us that these objects are simply scalars and μ can be treated as some internal index. This also means that Poincaré invariance is an internal symmetry at fixed τ, σ.

[10] don't worry about this for now.

[11] I suggest the reader to see [3]