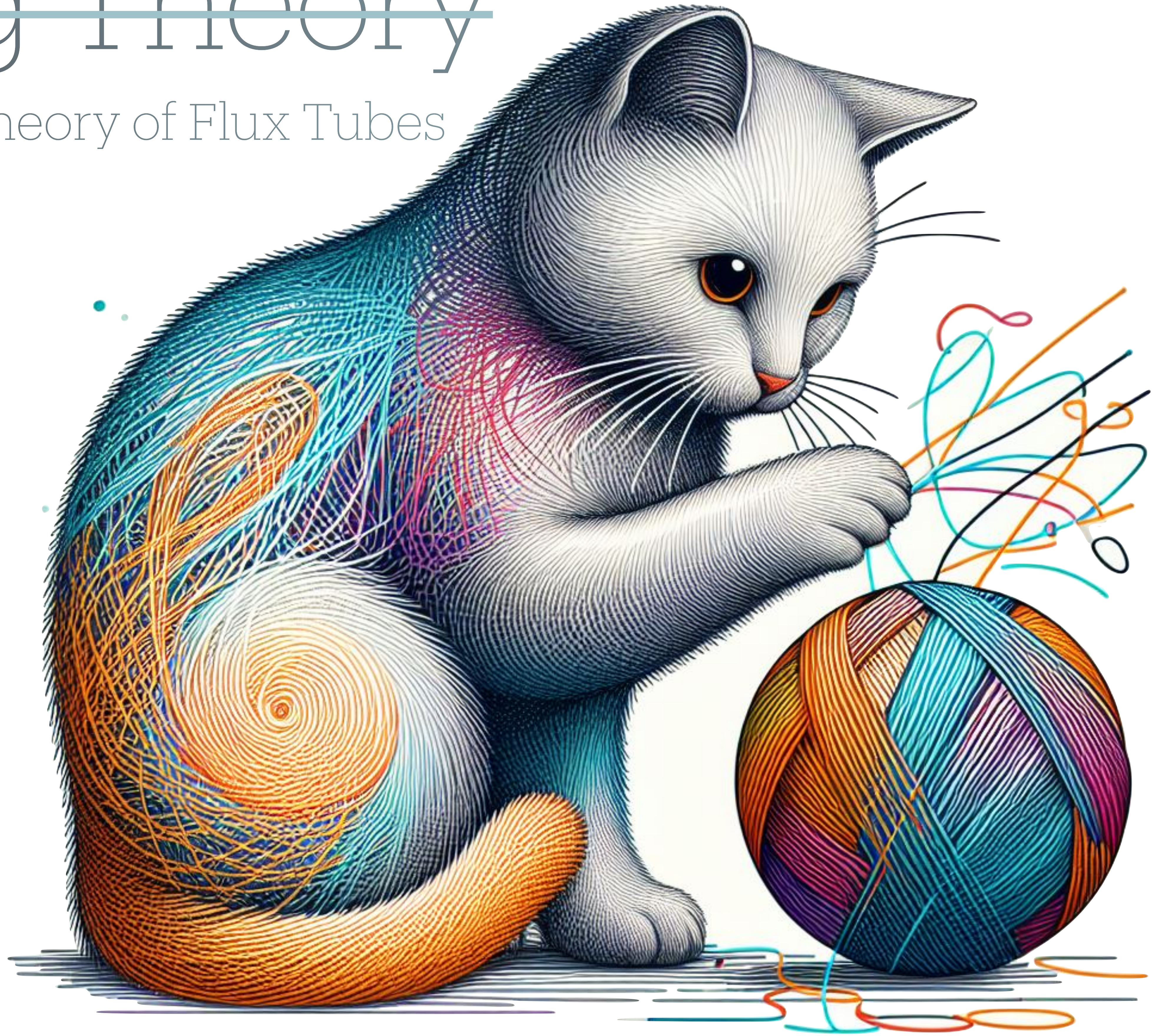
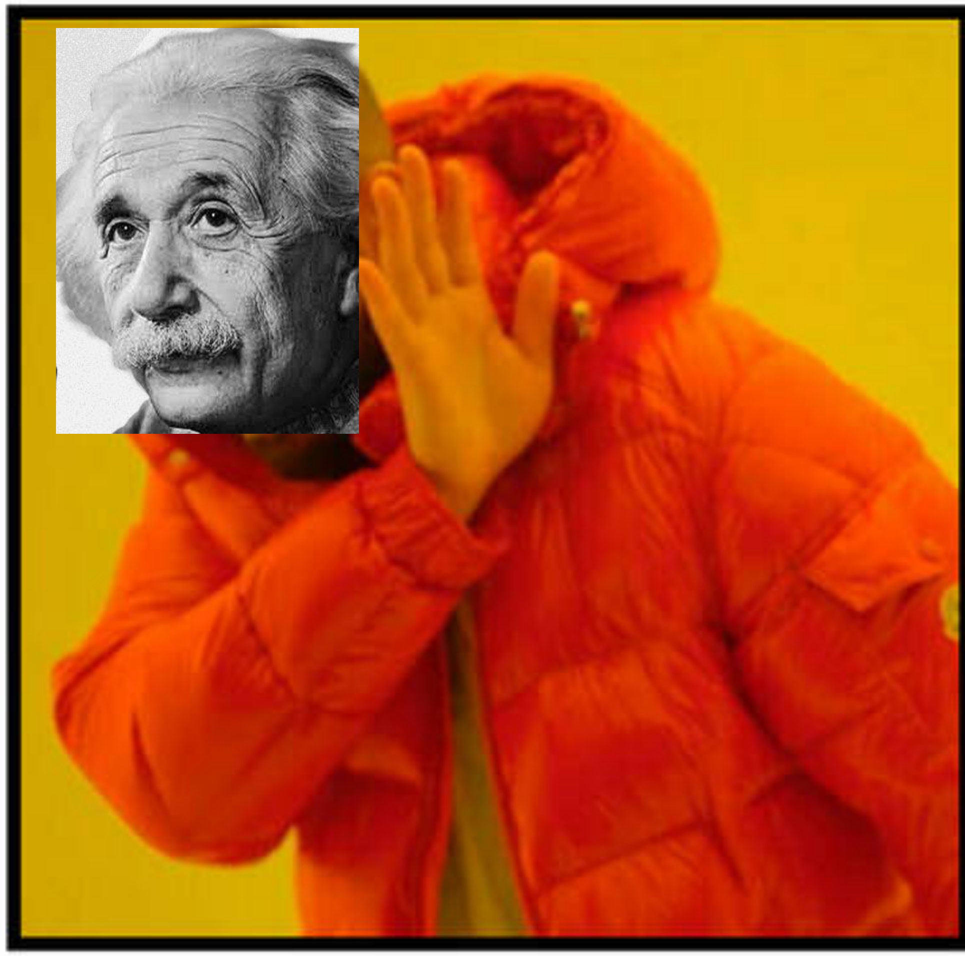
~~String Theory~~

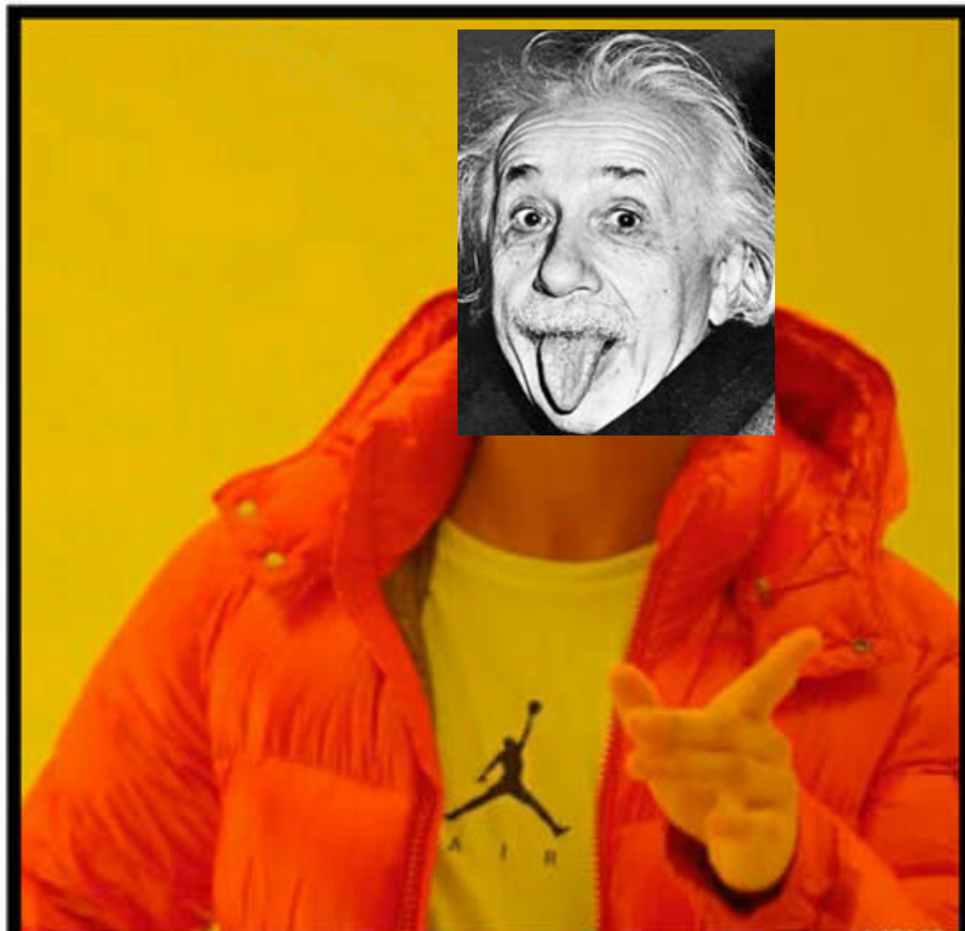
A Quantum Theory of Flux Tubes



Bhavay Tyagi
April 29th 2024



String Theory



A Quantum Theory of Flux Tubes

A Brief History*

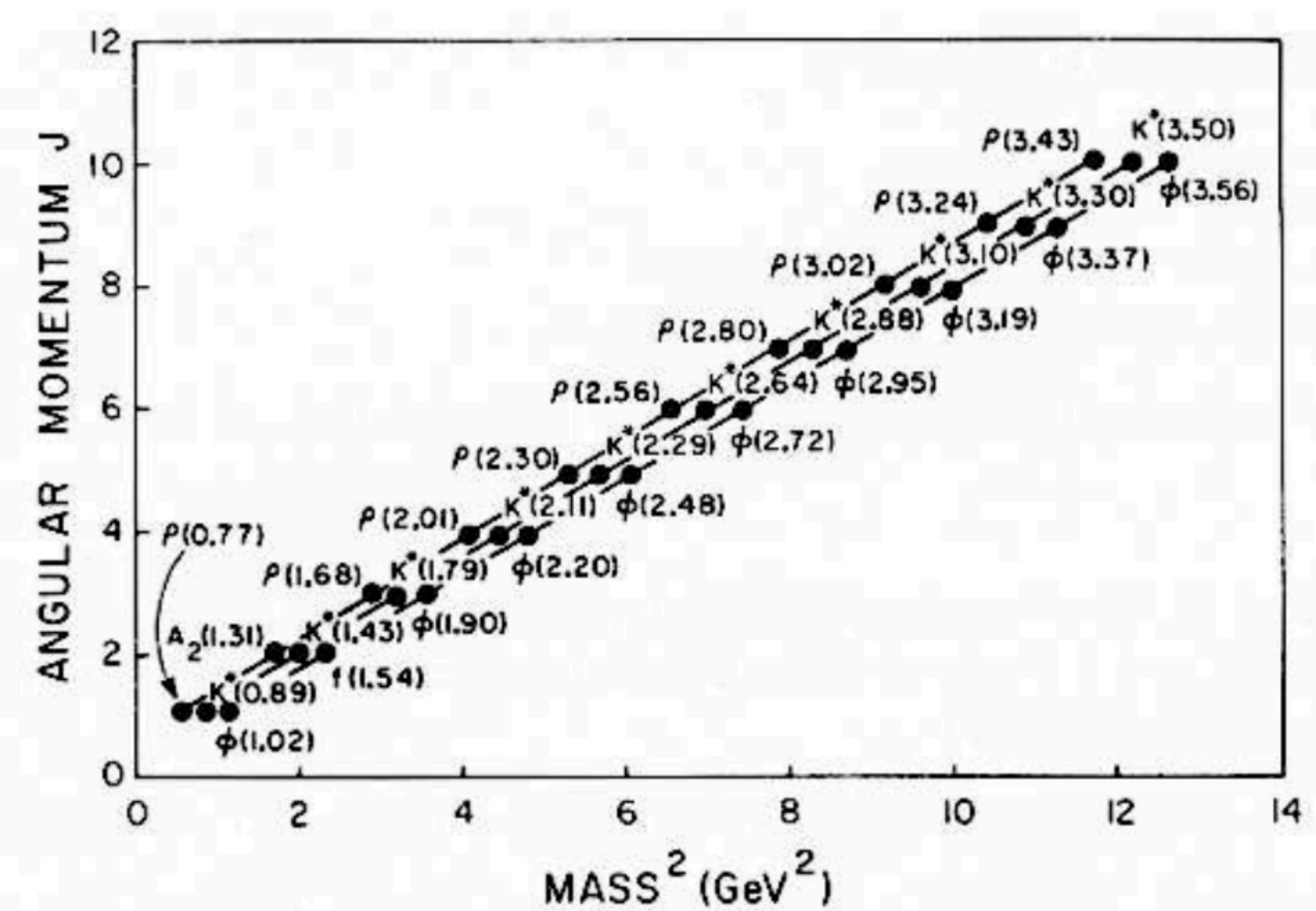
Of Problems in Hadronic Physics

pre-QCD (1960s-70s): Proliferation of strongly interacting confined states (hadrons). Especially resonances with exceptionally high spin.

$J = \alpha' m^2 + \alpha(0)$ Regge Trajectory

Universal Regge Slope

$\alpha' \sim 1(GeV)^{-2}$



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Universal Regge Slope

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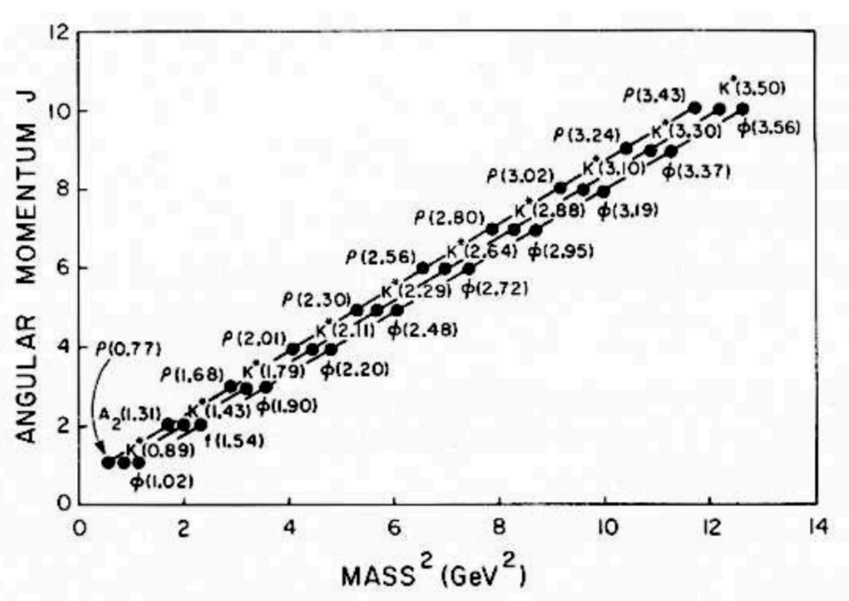
- Huge number of resonances meant they weren't fundamental.
- QFTs: spin-0, spin-1/2 and spin-1 (scalar, YM etc.)
- Robust framework for weak interactions but failed when strong interactions were naively treated.



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pre-QCD (1960s-70s): Another apparent puzzle was the high energy behaviour of scattering amplitudes.

A Brief History*

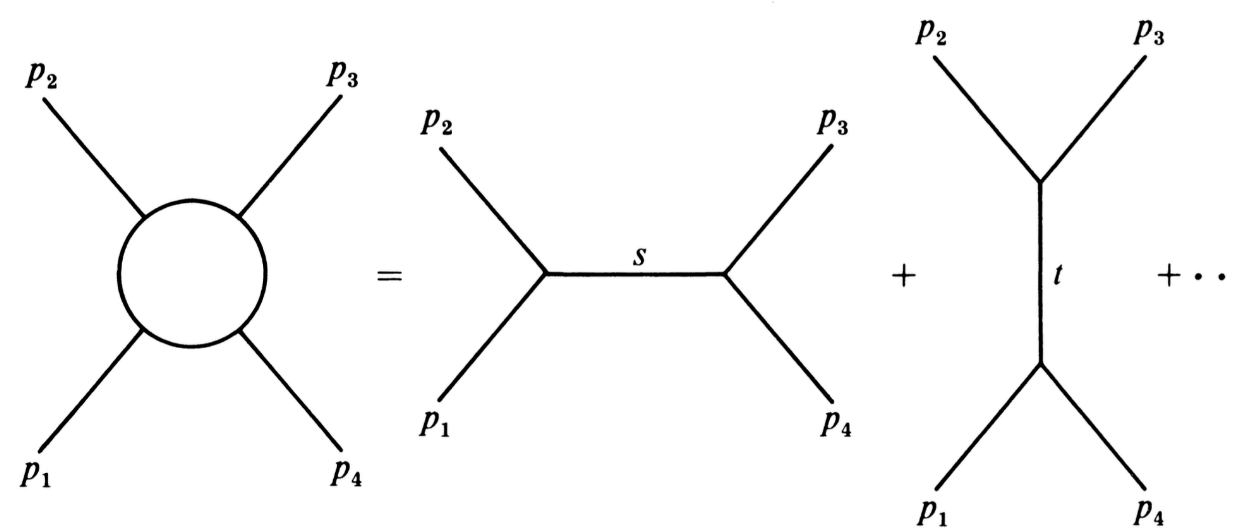
Of Problems in Hadronic Physics

pre-QCD (1960s-70s): Another apparent puzzle was the high energy behaviour of scattering amplitudes.

- Consider elastic scattering. Incoming (p_1, p_2) and outgoing (p_3, p_4) momenta. $(m^2 = -p^2)^*$

$$s = -(p_1 + p_2)^2,\ t = -(p_2 + p_3)^2,\ u = -(p_1 + p_3)^2$$

$$s + t + u = \sum_i m_i^2$$



Consider a term in the scattering matrix $A \propto tr(\lambda_1 \lambda_2 \lambda_3 \lambda_4)$, where λ_i is the flavour matrix.

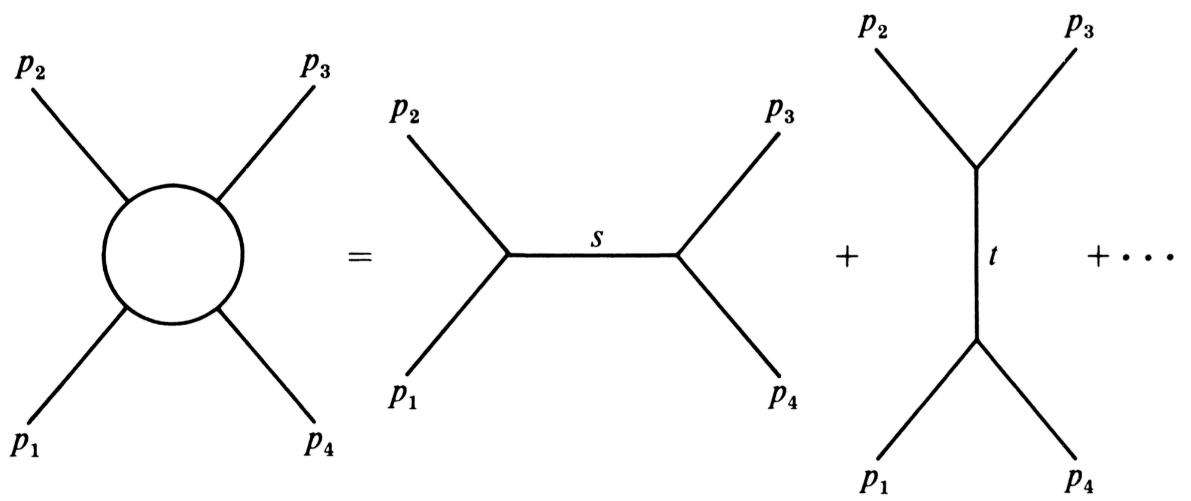
- This is invariant under cyclic $1234 \to 2341$ and hence $p_1 p_2 p_3 p_4 \to p_2 p_3 p_4 p_1$

$s \leftrightarrow t$ should then be a symmetry of the scattering amplitude $A(s, t)$.

*diag$(-++\ldots+)$ (The Better Metric)

A Brief History*

Of Problems in Hadronic Physics



- Consider $\phi^*\phi\sigma$ where σ is spin-0.
- T-channel: $A(s,t) = -g^2/(t-M^2)$ vanishes for $t \to \infty$ FANTASTIC!
- If $\sigma_{\mu_1\mu_2\ldots\mu_J}$ is now spin-J; The coupling looks like

  $\phi^*\partial_{\mu_1}\partial_{\mu_2}\ldots\partial_{\mu_J}\phi\,.\,\sigma^{\mu_1\mu_2\ldots\mu_J}$

- If the ϕ are scalars then the t-channel amplitude for high energy*

$$A_J(s,t) = -g^2\frac{(-s)^J}{t-M^2}$$

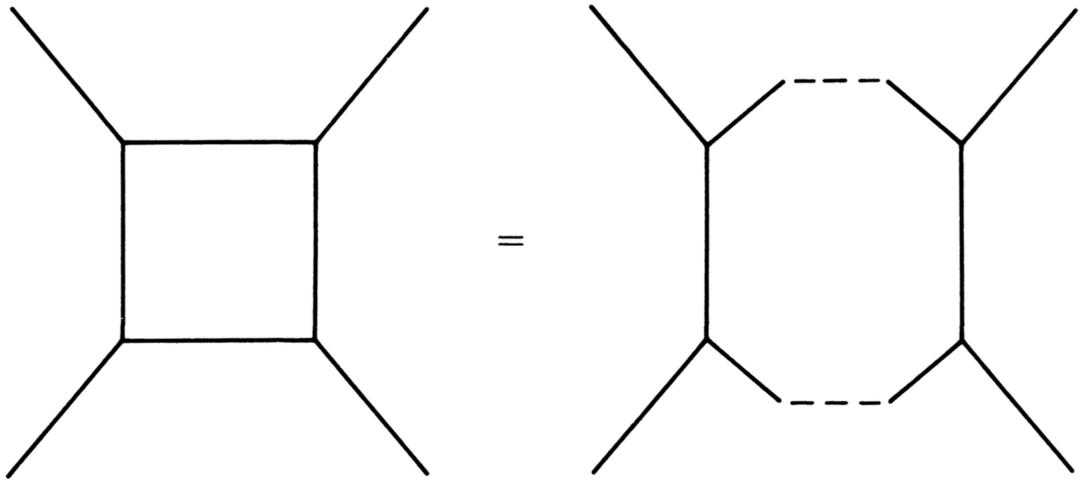
Divergent for Higher Spins

Challenge: Build QFTs where the exchange of high-spin particles is not divergent at high energies in tree level diagrams.

*This is the behaviour of the tree-level scattering amplitude in the asymptotic region of large s, fixed t. The exact formula (for moderate s) is more complicated, involving a Legendre polynomial. We prefer to write only the high-energy behaviour, which is transparent and adequate for our purposes.

A Brief History*

Of Problems in Hadronic Physics



- Recap: $s \leftrightarrow t$ and Divergent for high-J
- Q: Try sewing the tree-level diagrams?
- Loop integrand in n dimensions $\sim \int d^n p \frac{A^2}{(p^2)^2}$
- In 4D this integral is well behaved for $J < 1$ and unrenormalizable for $J > 1$.
- For various masses and spins: $$A(s,t) = -\sum_J \frac{g_J^2(-s)^J}{t - M_J^2}$$

A Brief History*

Of Problems in Hadronic Physics

$$A(s,t) = -\sum_J \frac{g_J^2(-s)^J}{t - M_J^2}$$

- Is this a finite sum?

- If it is then the "most" dominant behaviour should come from the term with highest-J. Is there a Hadron of highest-J? No!

- In nature the high energy behaviour of hadron scattering amplitudes is much softer than any individual term in this series. More like $A(s,t) = -g^2/(t - M^2)$

- Is this an infinite sum then?

- The high energy behaviour is captured by the whole sum better than any individual term in the series*.

*e^{-x} is small for $x \to \infty$ than any individual term in the power series expansion.

A Brief History*

Of Problems in Hadronic Physics

$$A(s,t) = -\sum_J \frac{g_J^2(-s)^J}{t - M_J^2}$$

- $A(s,t)$ should have both s-channel resonances and t-channel poles.
- A finite sum, for fixed t, implies no s-channel poles $\implies$ only a function of s.
- Due to this reason in perturbative QFTs we satisfy this crossing symmetry $(s \leftrightarrow t)$ by including "all" diagrams.
- The point is we have to accept that this is an infinite sum, with fixed t, and might diverge for some s values giving s-channel poles.

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Dolen, Horn and Schmid (1968): This is indeed true.

A Brief History*

Of Problems in Hadronic Physics

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- Therefore, we don't have to include s-channel diagrams altogether.
- Equally valid if we start from $A'(s,t) = -\sum_J \frac{g_J^2(-t)^J}{s - M_J^2}$
- Entire Amplitude can be expressed as a sum of s-channel **or** t-channel diagrams.
- First evidence of "Dualities" in physics. S- and t-channel give a dual description of the same physics.
- Is "Duality" an approximation or a Principle?
- How to choose masses and couplings to show $A(s,t) = A'(s,t)$?

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Gabriele Veneziano (1968): $A(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s) - \alpha(t))}$

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Gabriele Veneziano (1968): $A(s,t) = \frac{\Gamma(-\alpha(s))\Gamma(-\alpha(t))}{\Gamma(-\alpha(s)-\alpha(t))}$

- $\Gamma(u) = \int_0^\infty t^{u-1}e^{-t}dt$ (Euler Gamma Function)
- $A(s,t) = -\sum_{n=0}^{\infty} \frac{(\alpha(s)+1)(\alpha(s)+2)\dots(\alpha(s)+n)}{n!}\frac{1}{\alpha(t)-n} \quad s \leftrightarrow t$
- For Regge trajectory $\alpha(t) = \alpha(0) + \alpha'(t)t$, the singularities above are simple poles in the t-channel corresponding to $M^2 = (n - \alpha(o))/\alpha'$ where $n = 0,1,2,...$
- Therefore the smallest possible mass of particle with spin-J is given by $(J - \alpha(0))/\alpha'$. This is also indicates particles of mass M^2 lie on the Regge trajectory.
- For any QFT to be valid the residues of poles must be positive and that is not manifest in the above formula. (No-Ghost Theorem). The above formula does not tell us anything about the high energy behaviour.
- This Veneziano amplitude was an ad hoc way to deal with this "crossing-symmetry". The urge to understand this in different asymptotic regions led to the birth of Regge Pole Theory, Dual Models and String Theory.

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↓

Igor Klebanov, J. Polchinski, Strominger (2000s): Effective String Theory

Here are the problems:

1. Are all the (measurable) dimensionless parameters that characterize the physical universe calculable in principle or are some merely determined by historical or quantum mechanical accident and uncalculable?
   David Gross, Institute for Theoretical Physics, University of California, Santa Barbara

2. How can quantum gravity help explain the origin of the universe?
   Edward Witten, California Institute of Technology and Institute for Advanced Study, Princeton

3. What is the lifetime of the proton and how do we understand it?
   Steve Gubser, Princeton University and California Institute of Technology

4. Is Nature supersymmetric, and if so, how is supersymmetry broken?
   Sergio Ferrara, CERN (European Laboratory of Particle Physics)
   Gordon Kane, University of Michigan

5. Why does the universe appear to have one time and three space dimensions?
   Shamit Kachru, University of California, Berkeley
   Sunil Mukhi, Tata Institute of Fundamental Research
   Hiroshi Ooguri, California Institute of Technology

6. Why does the cosmological constant have the value that it has, is it zero and is it really constant?
   Andrew Chamblin, Massachusetts Institute of Technology
   Renata Kallosh, Stanford University

7. What are the fundamental degrees of freedom of M-theory (the theory whose low-energy limit is eleven-dimensional supergravity and which subsumes the five consistent superstring theories) and does the theory describe Nature?
   Louise Dolan, University of North Carolina, Chapel Hill
   Annamaria Sinkovics, Spinoza Institute
   Billy & Linda Rose, San Antonio College

8. What is the resolution of the black hole information paradox?
   Tibra Ali, Department of Applied Mathematics and Theoretical Physics, Cambridge
   Samir Mathur, Ohio State University

9. What physics explains the enormous disparity between the gravitational scale and the typical mass scale of the elementary particles?
   Matt Strassler, Institute for Advanced Study, Princeton

10. Can we quantitatively understand quark and gluon confinement in Quantum Chromodynamics and the existence of a mass gap?
    Igor Klebanov, Princeton University
    Oyvind Tafjord, McGill University

Strings 2000: UMich Ann Arbor

Some Heuristic Arguments

A Hint in QCD

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\sum_{q=1}^{N_f}\bar{\psi}_q^i(\not{D}_\mu)_{ij}\psi_q^j - \sum_{q=1}^{N_f} m_q\bar{\psi}_q^i\psi_q^i$$

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$$

$$(D_\mu)_{ij} \equiv \delta_{ij}\partial_\mu + ig\frac{\lambda_{ij}^a}{2}A_\mu^a$$

$$A_\mu \equiv A_\mu^a\frac{\lambda_a}{2}, \quad a = 1, ..., 8$$

- The QCD Lagrangian →
- The Beta function equation → $\frac{d\alpha_s(\mu)}{d\log\mu^2} \equiv \beta(\alpha_s) = -\alpha_s\left(\beta_0(\frac{\alpha_s}{4\pi}) + \beta_1(\frac{\alpha_s}{4\pi})^2 + ...\right)$
- $\alpha_s \equiv g^2/4\pi$ and $\beta_0 = 11 - \frac{2}{3}N_f$, can only do perturbation theory when $\mu >> \Lambda_{\text{QCD}}$.
- Bound-quark-states are found at 1fm or Λ^{-1} aka Avg. Hadronic Size.

Some Heuristic Arguments

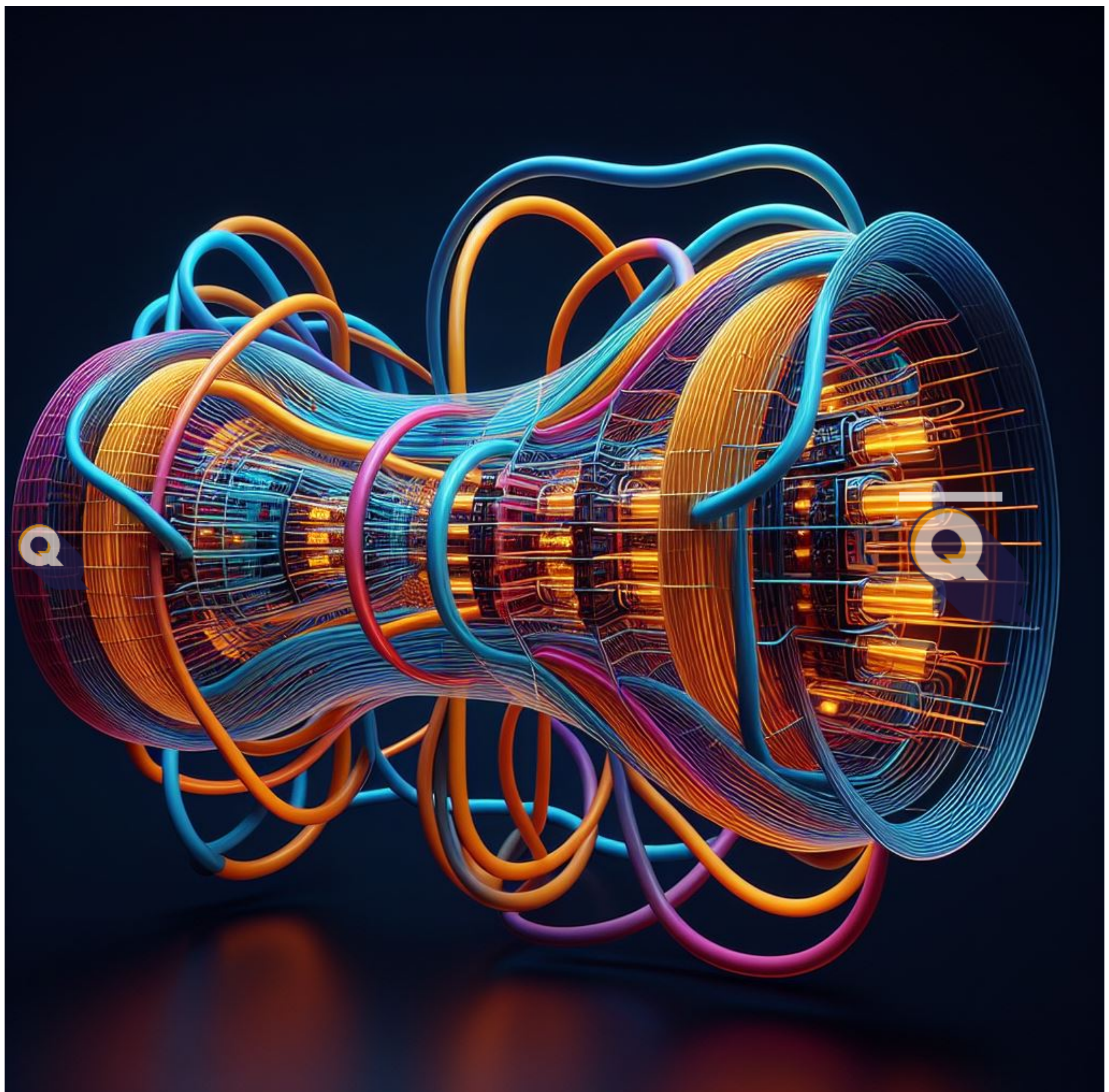
A Hint in QCD

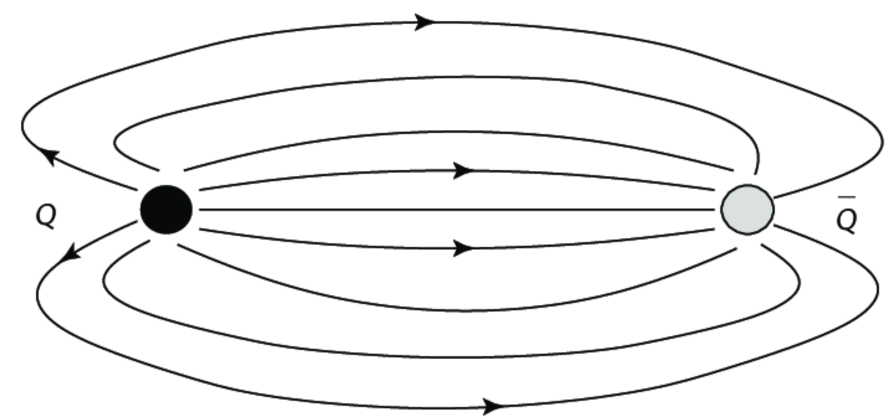
- Due to confinement, quark masses aren't physical or can't be directly measured.
- Light mesons and baryons obey the Regge trajectory.
- Free quarks have never been detected means that the interaction between them must be strong at large length scales and a $q\bar{q}$-pair is created when the quarks are significantly far away (even cosmic scales?).
- At 1GeV they appear in the Hadronic state, this suggests a linear energy density between a quark and an anti-quark of the order $T = \frac{\Delta E}{\Delta r} \simeq 1\frac{GeV}{1fm} = 0.2\ GeV^2$.
- At short distances < 1fm the quark-antiquark potential is Coulombic ($\sim 1/r$). This is called Asymptotic Freedom. At large distances, field lines confine themselves into a chromoelectric flux tube.
- If the tubes are longer than they're thick we can describe them using 1 dimensional strings.
- A semi-classical treatment of these strings gives the potential $V(r) = Tr + \mu + \gamma/r + \mathcal{O}(1/r^2)$
- A string-like object already exists in QCD.

Some Heuristic Arguments

A Hint in QCD

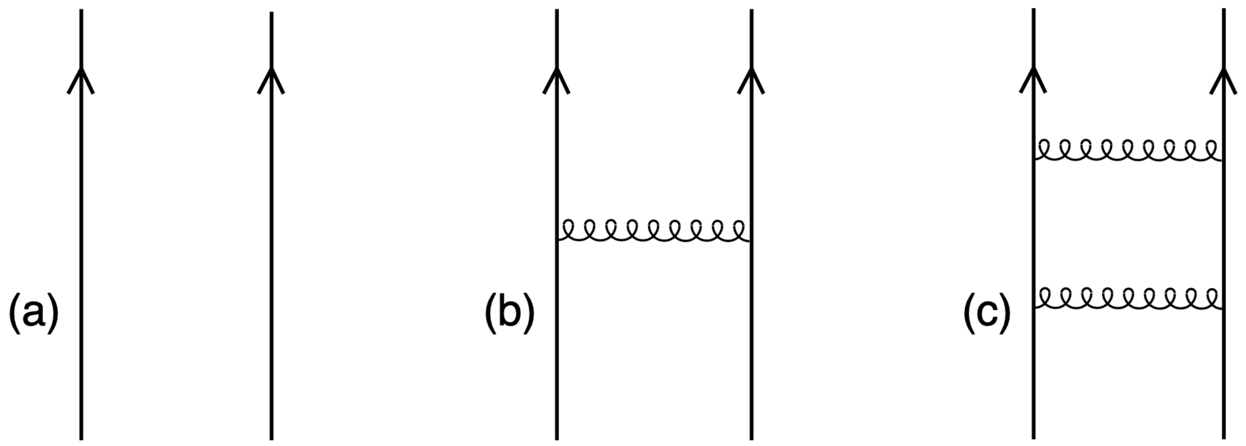
Cool QCD Flux Tube Image That Drives People Crazy





A Quick Comment on

Quantum Gravity



- One graviton exchange $\propto G_N$
- The Ratio of (b) and (a) is the dimensionless combination $G_N E^2 \hbar^{-1} c^{-5}$ (only combination possible).
- For $\hbar = c = 1$ we can define Planck Mass $M_P = G_N^{-1/2}$=1.22×10^{19} GeV. $M_p^{-1} = 1.6 \times 10^{-33}$cm. Therefore the Ratio above is given by $(E/M_p)^2$.
- This is an irrelevant coupling.
- For two graviton exchange we sum over intermediate states of energy E', then the ratio with zero-graviton exchange is $G_N^2 E^2 \int dE' E'$
- Diverges at high energy.

A Quick Comment on

Quantum Gravity

- Such a bad UV behaviour is seen when we try to quantize gravity.
- In General Relativity the gravitational field is a massless spin-2 field called the Graviton field.
- Interactions are governed by the non-abelian local symmetry group called Diffeomorphisms of spacetime.
- For a t-channel exchange of a spin-1 particle $A_{\text{YM}} \sim s/t$ is barely renormalizable. For QG in 4 dimensions $A_{\text{QG}} \sim s^2/t$ hopelessly unrenormalizable.

String Theory In a Few Minutes

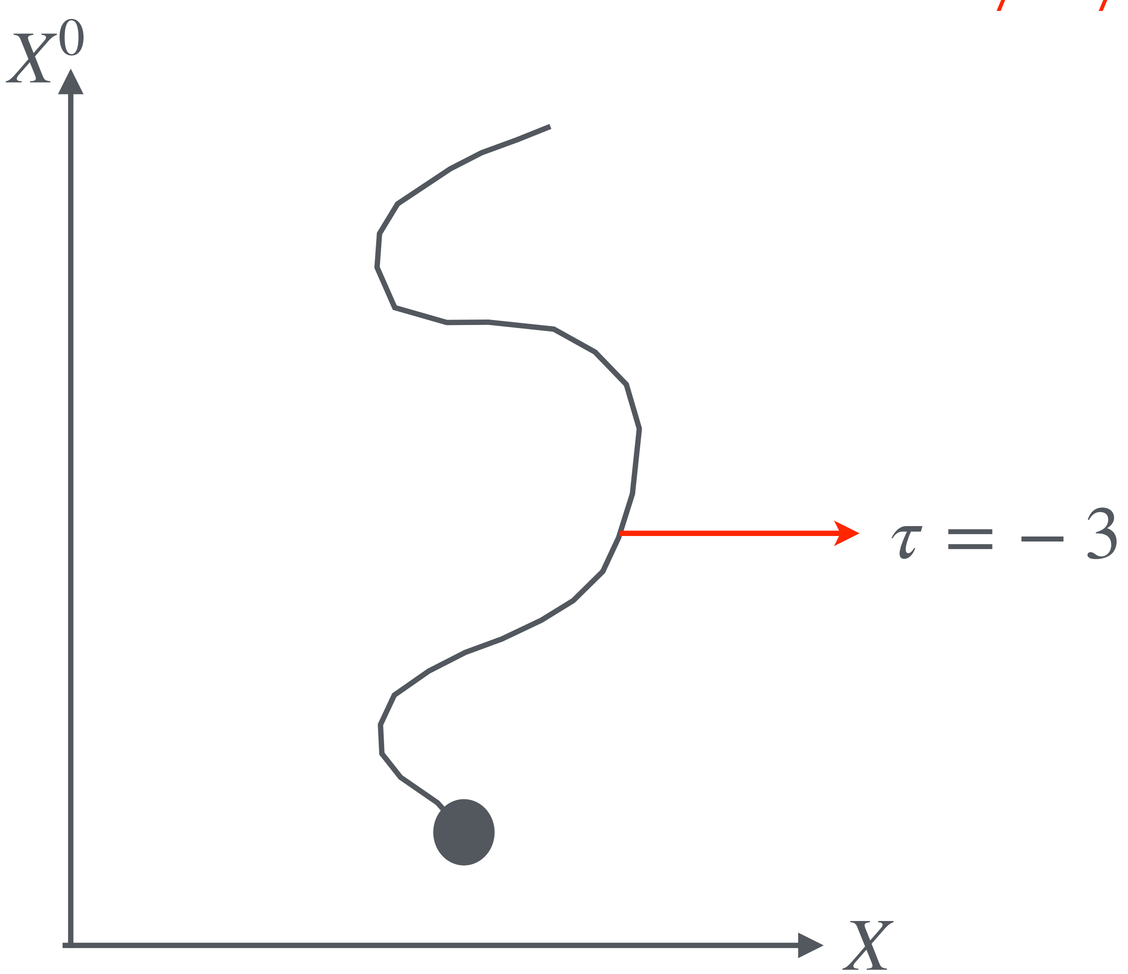
A lightening review

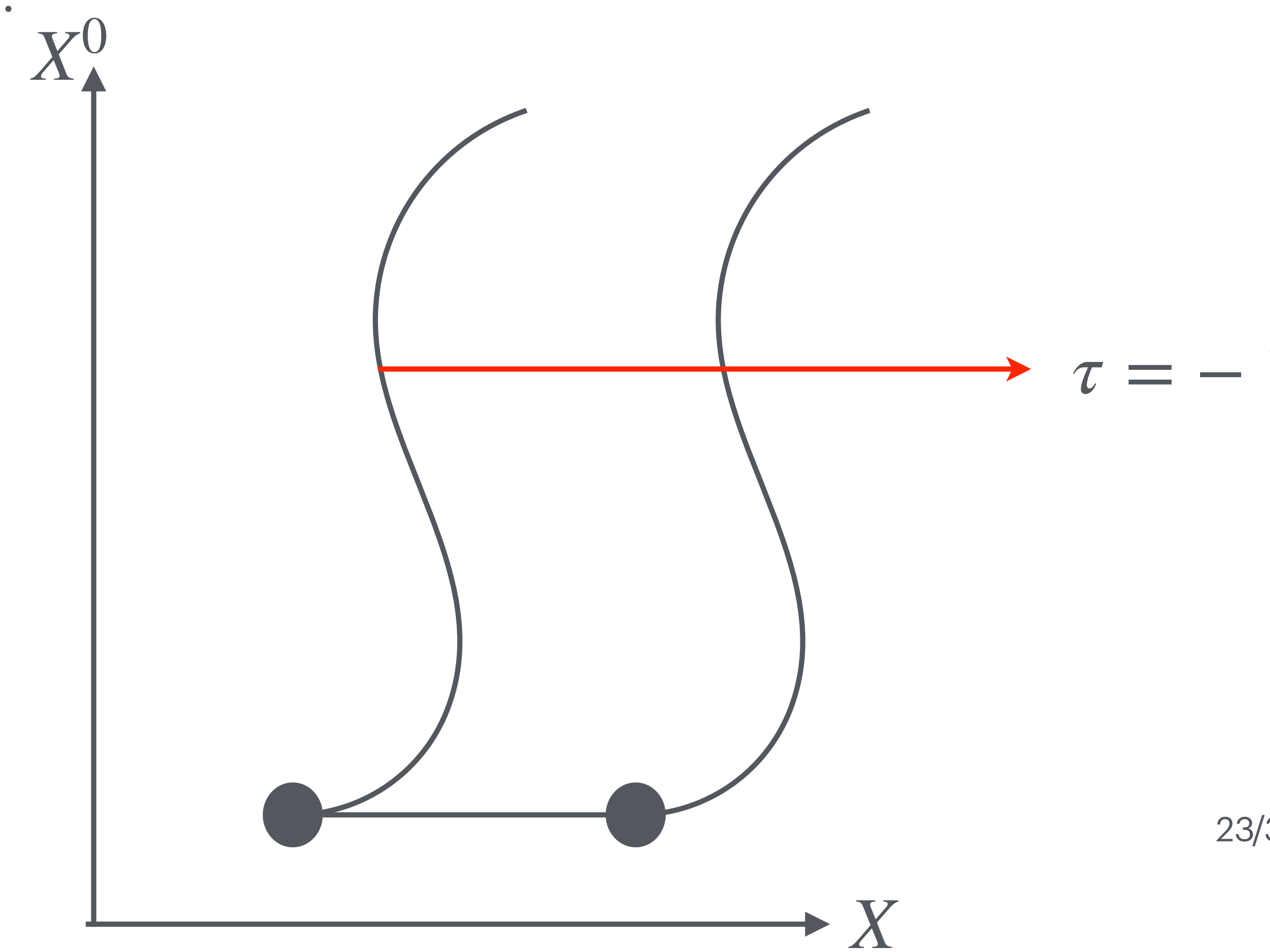
- Massless spin-2 state whose interactions at low energies reduces to GR
- Perturbative QG Theory
- “Grand Unification”
- Extra Dimensions
- “Supersymmetry”
- Chiral Gauge Coupling
- No Free Parameters
- Uniqueness

String Theory In a Few Minutes

A lightening review

- Let our fundamental object be a 1-dimensional mathematical curve.
- Choose space like coordinate $\sigma \in [0,\pi]$ and a timeline parameter τ to describe evolutionary dynamics.





String Theory In a Few Minutes

A lightening review

- Let our fundamental object be a 1-dimensional mathematical curve.
- Choose space like coordinate $\sigma \in [0,\pi]$ and a timeline parameter τ to describe evolutionary dynamics.
- In D flat spacetime dimensions with metric $\eta_{\mu\nu} = diag(-,+,+,\ldots,+)$ the string looks like $X^\mu(\tau,\sigma)$.
- Naturally the action describing the string should be free from the parameters and should depend on the embedding in spacetime.
- For point-particles we extremize the world-line so for strings we extremize the world-sheet.

String Theory In a Few Minutes

The Open String Spectrum

- The most general diff x Weyl invariant action
  $$S_p = -\int_M d\tau d\sigma (-\gamma)^{1/2}\left(\frac{1}{4\pi\alpha'}\gamma^{ab}\partial_a X^\mu \partial_b X_\mu + \frac{\lambda}{4\pi}R\right)$$
- In the light-cone coordinates
  $$x^{\pm} = 2^{-1/2}(x^0 \pm x^1),\;\; x^i,\; i = 2,\ldots,D-1$$
- Set
  $\tau = x^+, p^- =$ Energy, (x^-, p^+) – logitudnal and (x^i, p^i) – transverse are spatial coordinates.
- For open strings $-\infty \leq \tau \leq \infty$ and $0 \leq \sigma \leq l$.
- The light-cone gauge
  $$X^+ = \tau$$
  $$\partial_\sigma \gamma_{\sigma\sigma} = 0$$
  $$det\gamma_{ab} = -1$$
- Lagrangian
  $$L = -\frac{l}{2\pi\alpha'}\gamma_{\sigma\sigma}\partial_\tau x^- + \frac{1}{4\pi\alpha'}\int_0^l d\sigma\left(\gamma_{\sigma\sigma}\partial_\tau X^i \partial_\tau X^i - \gamma_{\sigma\sigma}^{-1}\partial_\sigma X^i \partial_\sigma X^i\right)$$
- Hamiltonian
- Momentum conjugate to $X^i(\tau,\sigma)$ is
  $$\Pi^i = \frac{\delta L}{\delta(\partial_\tau X^i)} = \frac{1}{2\pi\alpha'}\gamma_{\sigma\sigma}\partial_\tau X^i = \frac{p^+}{l}\partial_\tau X^i$$

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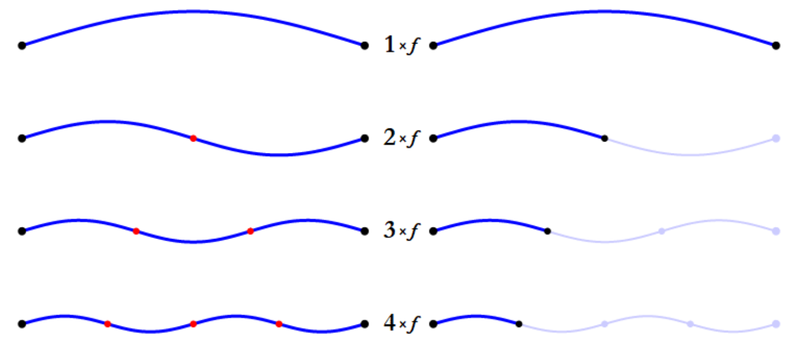
String Theory In a Few Minutes

The Open String Spectrum



- Modes for each m and i satisfy
  $\alpha^i_m \sim m^{1/2} a, \;\; \alpha^i_{-m} \sim m^{1/2} a^\dagger, \;\; m > 0 \;\; [a, a^\dagger] = 1$
- The Oscillator is labelled by the direction of oscillation i and the harmonic m.
- State $|0; k\rangle$ is annihilated by the lowering operator and will be an eigenstate of the COM momenta
  $p^+|0'k\rangle = k^+|0;k\rangle, \;\; p^i|0;k\rangle = k^i|0;k\rangle$
  $\alpha^i_m|0;k\rangle = 0, \, m > 0$

- A general state is built from raising operators
  $$|N;k\rangle = \left[\prod_{i=2}^{D-1}\prod_{n=1}^{\infty} \frac{(\alpha^i_{-n})^{N_{in}}}{(n^{N_{in}} N_{in}!)^{1/2}}\right]|0;k\rangle$$
- States are labelled by COM momenta $k = (k^+, k^i)$ and occupation number N_{in} for each mode.
- COM momenta is the degree of freedom for the point particle and the oscillator part is the internal degree of freedom.
- Every choice of N_{in} represents a different spin state.
- The lowest excited states of the string are obtained by exciting one of the n=1 modes once
  $\alpha^i_{-1}|0;k\rangle, \;\; m^2 = \dfrac{26 - D}{24\alpha'}$

*backup slide at 36 and 37

String Theory In a Few Minutes

The Open String Spectrum

- In D dimensions, a massless vector has $D-2$ spin states. For $n=1$ we get $D-2$ spin states implies that they must be massless. Which means $D=26$.

- Quantum Mechanical spectrum is Lorentz invariant for a specific number of spacetime dimensions.

- Theory is classically invariant under any D but when quantised there is an anomaly except when $D=26$. At level N, the maximum eigenvalue of a given spin component, is N. For some fixed spin l, $l \leq \frac{D-2}{24} + \alpha' m^2$, where α' is the "Regge Slope".

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↓

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↓

J. Maldacena(1997): Theories with gravity in D dimensions are dual to large-N field theories in D-1 dimensions.

Coming Full Circle

AdS/CFT

J. Maldacena(1997): Theories with gravity in D dimensions are dual to large-N field theories in D-1 dimensions.

- Our love for expanding about a small parameter trumps everything.
- t'Hooft limit allows this for strongly coupled systems. $SU(N)$ when $N \to \infty$.
- Strings move on a spacetime which has boundary at spatial infinity (AdS). So a light-ray can travel to the boundary and return in finite time. Massive particles can never reach this boundary.
- The radius of curvature depends on N. So in the $N \to \infty$ limit, the curvature is small (Asymptotically AdS).
- String theory has gravity automatically built into it.
- Since the field theory lives in 1 lower dimension it can be assumed that it lives on the boundary of this AdS space.
- Notice when gravitational theories are being treated at high energies they become topological since there exits an integral over the metric. So the metric dependence goes away.
- We always talk about finite energy excitations, so we're summing over all spacetimes.

Coming Full Circle

Large-N

- Counterintuitive at first glance
- Remember fields are related to each other by symmetries so the combined behaviour gets constrained when you add more fields.
- Consider $S_{YM} = -\frac{1}{2g^2}\int d^4x \; tr \; F^{\mu\nu}F_{\mu\nu}$
- We introduce cut-off
  $\Lambda_{QCD} = \Lambda_{UV} \exp\left\{-\frac{3}{22}\frac{(4\pi^2)}{(g^2)N}\right\}$
- Taking $N \to \infty$ keeping g^2 fixed, results in no parametric separation between the theories.

- Define t'Hooft coupling $\lambda = g^2 N$ and now for fixed cut-off and λ, take $N \to \infty$
- $S_{YM} = -\frac{N}{2\lambda}\int d^4x \; tr \; F^{\mu\nu}F_{\mu\nu}$

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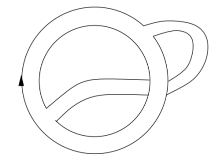
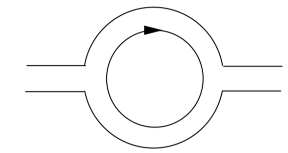
Coming Full Circle

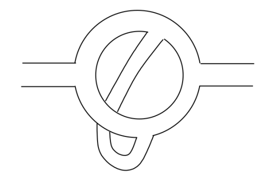
Large-N Diagrams



$\sim \left(\frac{\lambda}{N}\right)^3 \left(\frac{N}{\lambda}\right)^2 N \sim \lambda$

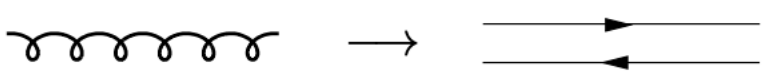


$\sim \left(\frac{\lambda}{N}\right)^4 \left(\frac{N}{\lambda}\right)^2 N \sim \frac{\lambda^2}{N}$



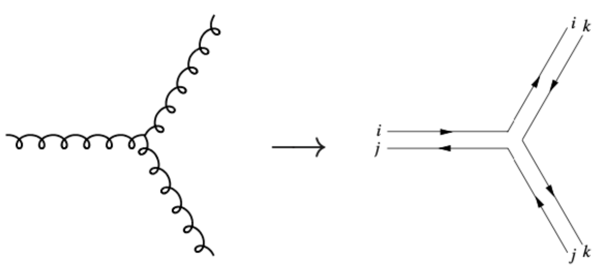
$\sim \left(\frac{\lambda}{N}\right)^7 \left(\frac{N}{\lambda}\right)^4 \sim \frac{\lambda^3}{N^3}$

- Double line propagator

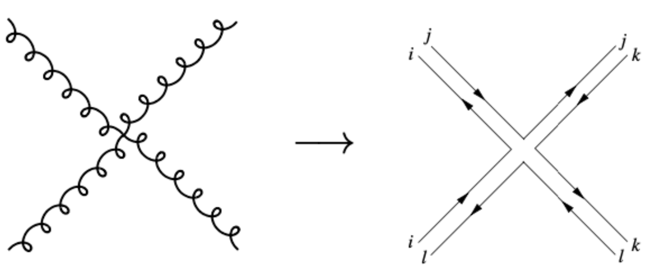


$\sim \frac{\lambda}{N}$

- Cubic and Quartic vertex



$\sim \frac{N}{\lambda}$

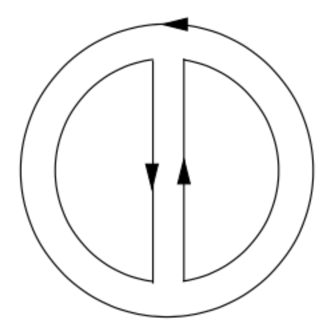


$\sim \frac{N}{\lambda}$

- Generally any diagram

$$\text{diag} \sim \left(\frac{\lambda}{N}\right)^{\#\text{propagators}} \left(\frac{N}{\lambda}\right)^{\#\text{vertices}} N^{\#\text{index contractions}}$$

- Vacuum Bubble



$\sim \left(\frac{\lambda}{N}\right)^3 \left(\frac{N}{\lambda}\right)^2 N^3 \sim \lambda N^2$

- Only "Planar Diagrams" contribute.

*backup slide at 39

Conclusion

Engineering the world we see

- String like objects have always existed in the physics of Hadrons. At least to some approximation.
- The reproduction of the Regge slope in Bosonic String Theory was one of the first hints that String Theory could be "the" Theory.
- However it comes with its own challenges.
- One of the main problems is to deal with extra dimensions. The number ways to compactify the extra 6 dimensions (Calabi-Yau Manifolds) to give our 4D world is something like 10^{500} "plausible" configurations. Each leading to different laws of physics.
- Three days ago an Article on Quanta Magazine has shed light on this that how physicists are now training AIs to sought through these compactifications using hints from neural networks.

Conclusion

Engineering the world we see

- The final point is AdS/CFT provides a way to use this the $1 \leftrightarrow 1$ mapping between a gauge theory on the boundary and a "string theory" in the bulk to even try to attempt to find a fundamental theory of "everything".
- Whether it is right or not, this cross-fertilisation between different fields and to serve as a "language" through which experts from different fields can communicate is the real power of String Theory.

Thank you for you attention.

References

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[5] QCD AND STRING THEORY IGOR R. KLEBANOV

[6] The Large N Limit of Superconformal Field Theories and Supergravity J. Maldacena

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String Theory In a Few Minutes

The Open String Spectrum

- $H = \frac{l}{4\pi\alpha' p^+}\int_0^l d\sigma \left(2\pi\alpha'\Pi^i\Pi^i + \frac{1}{2\pi\alpha'}\partial_\sigma X^i \partial_\sigma X^i\right)$
- Boundary condition
  $\partial_\sigma X^i = 0$ at $\sigma = 0, l$
- Using Hamilton's EOM we have the wave-equation
  $\partial_\tau^2 X^i = c^2 \partial_\sigma^2 X^i$
- Solution:
  $X^i(\tau,\sigma) = x^i + \frac{p^i}{p^+}\tau + i(2\alpha')^{1/2}\sum_{n=-\infty, n\neq 0}^{\infty}\frac{1}{n}\alpha_n^i \exp\left\{-\frac{\pi i n c\tau}{l}\right\}\cos\left(\frac{\pi n \sigma}{l}\right)$
- Imposing equal time commutation relations
  $[x^-, p^+] = -i$
  $[X^i(\sigma), \Pi^i(\sigma')] = i\delta^{ij}\delta(\sigma - \sigma')$
- In terms of Fourier components
  $[x^i, p^+] = i\delta^{ij}$
  $[\alpha_m^i, \alpha_n^j] = m\delta^{ij}\delta_{m,-n}$

String Theory In a Few Minutes

The Open String Spectrum

- These form a Hilbert space $\mathscr{H}_1$ of one open string. Therefore $|0;0\rangle$ is the ground state of a string with zero momentum.

- Remember this above expertise is to act within the space of states of the string and not to create or destroy strings.

- For n strings the full Hilbert space would be $\mathscr{H} = |vacuum\rangle \oplus \mathscr{H}_1 \oplus \mathscr{H}_2 \oplus \ldots$

- Finally the "mode-expanded" Hamiltonian is
$$H = \frac{p^i p^i}{2p^+} + \frac{1}{2p^+\alpha'}\left(\sum_{n=1}^{\infty} \alpha^i_{-n}\alpha^i_n + A\right)$$

- Let's look at the lightest string state $|0;k\rangle, m^2 = \dfrac{2-D}{24\alpha'}$. Notice for $D > 2,\ \ m^2 < 0$. This state is a tachyon.

String Theory In a Few Minutes

The Open String Spectrum

- Lets break this down.
- Lorentz invariance requires a specific value of D
- There is no rest from for massless particles. $p^\mu = (E, E, 0, \ldots, 0)$.
- $SO(D-2)$ acting on transverse directions leave p^μ invariant.
- Massless particles are labelled by helicity λ, which is the eigenvalue under the single $SO(2)$ generator.
- So for Lorentz invariance we only need one state. However CPT symmetry takes $\lambda \to -\lambda$ so that means two states when $\lambda \neq 0$

Coming Full Circle

Large-N Diagrams

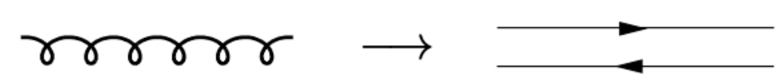
- Gluon Field $(A_\mu)^i_j, \;\; i,j = 1,\ldots,N$ is an $N \times N$ matrix.

- Propagator structure
  $\langle A^i_{\mu j}(x) A^k_{\nu l}(y) \rangle = \Delta_{\mu\nu}(x-y)\left(\delta^i_l \delta^k_l - \frac{1}{N}\delta^i_j \delta^k_l\right)$

- The $1/N$ factor comes from traceless $SU(N)$ gauge fields. Due to this suppression at leading order we can drop that term.

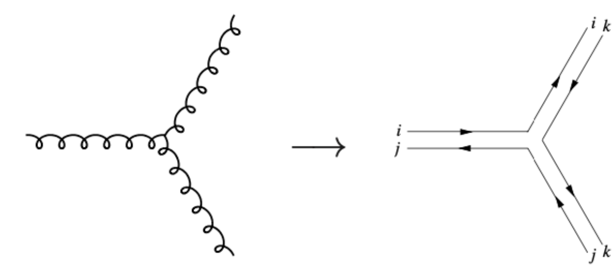
$$\langle A^i_{\mu j}(x) A^k_{\nu l}(y) \rangle = \Delta_{\mu\nu}(x-y)\delta^i_l \delta^k_l$$

- Double line propagator

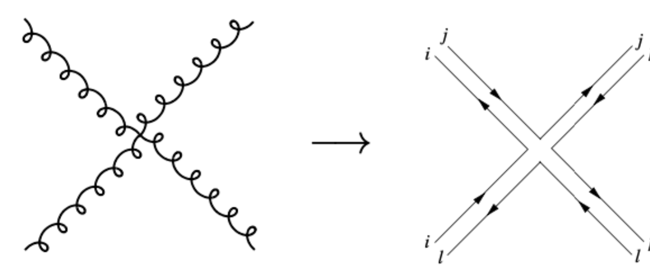


$\sim \frac{\lambda}{N}$

- Cubic and Quartic vertex



$\sim \frac{N}{\lambda}$



$\sim \frac{N}{\lambda}$

Coming Full Circle

Topology of Large-N Diagrams

- Planar Diagrams can be drawn on the surface of a sphere.
- A diagram can tile a 2-dim surface Σ using the following map:

$E = \#$ of Edges $= \#$ of propagators
$F = \#$ of faces $= \#$ of index loops
$V = \#$ of vertices

diag $\sim N^{F+V-E}\lambda^{E-V}$

- We can characterize any Riemann surface using the Euler Character
$\chi(\Sigma) = F + V - E = 2 - 2H$
where H is the genus.
- The point is that the sum of Feynman diagrams are weighted by their topology diag $\sim N^{\chi}\lambda^{E-V}$
- For each genus we can have a different tiling of the Riemann surface in the t′ Hooft expansion.